A Cauchy-Green Formula on the Unit Sphere in $\mathbf{C}^2$

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Abstract. In 1977 G. Henkin introduced an integral formula for solving $\overline{\partial}_b f = \mu$ where μ is a measure, on the boundary of a smooth strictly convex domain. This result is closely related to a "Cauchy-Green" formula on the sphere (see Chen and Shaw [**3**]). We give a direct elementary proof of the Cauchy-Green Theorem on the unit sphere and derive Henkin's solution of the $\bar{\partial}_b$ equation from this. We also give an application to an approximation result.

1. Introduction

Let Ω be a domain in the plane, with smooth boundary Γ. The classical Cauchy-Green formula states that for any $\phi \in C^1(\overline{\Omega})$ and $z \in \Omega$,

$$\phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\Omega} \frac{\partial \phi}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} \tag{1.1}$$

Note that the first term on the right of (1.1) is a holomorphic function Φ of z in the domain Ω. In fact, Φ extends continuously to $\overline{\Omega}$, and hence defines an element of the algebra $A(\overline{\Omega})$ consisting of functions holomorphic in Ω and continuous on $\overline{\Omega}$. Of course, if $\phi \in A(\overline{\Omega})$, (1.1) reduces to the Cauchy integral formula and $\Phi = \phi$.

The representation (1.1) has many applications in complex analysis. In the theory of approximation of continuous functions on a compact set $K \subset \mathbf{C}$ by rational functions with poles off K, one is led by considerations of duality to examine measures supported on K. The Cauchy transform of such a measure μ is defined by

$$\hat{\mu}(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z} \tag{1.2}$$

The integral defining $\hat{\mu}$ converges absolutely for almost all $z \in \mathbf{C}$. Using (1.1), one can easily show that for any smooth compactly supported function ϕ,

$$\int_K \phi(z)\, d\mu(z) = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{\partial \phi}{\partial \bar{z}}\, \hat{\mu}(z)\, d\bar{z} \wedge dz \tag{1.3}$$

That is, $\hat{\mu}$ satisfies the equation

$$\frac{\partial \hat{\mu}}{\partial \bar{z}} = -\pi \mu \tag{1.4}$$

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in the sense of distributions, and hence defines a holomorphic function on $\mathbf{C} \setminus K$. The Cauchy transform is a key tool in rational approximation theory in the plane.

We have been motivated by problems of rational approximation for subsets of the boundary S of the unit ball in $\mathbf{C}^2$. It is posible to do a kind of function theory on S analogous to the theory of analytic functions in the plane. The operator $\partial/\partial\bar{z}$ is replaced by the tangential Cauchy-Riemann operator

$$X = z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2}. \tag{1.5}$$

X is well-defined on $C^1(S)$ and for any relatively open subset Ω of S, annihilates the restrictions to Ω of functions holomorphic in a neighborhood of Ω in $\mathbf{C}^2$. The solutions to $X\phi = 0$ on Ω are known as CR functions on Ω. A good general reference for the theory of CR functions is the book [**2**].

One would like an analogue of the Cauchy transform for measures on S. Given a measure μ on S, G. Henkin in 1977 [**4**] constructed a function K_μ, summable with respect to three-dimensional Hausdorff measure $d\sigma$ on S, satisfying

$$\bar{\partial}_b K_\mu = -2\pi^2 \mu \tag{1.6}$$

in the sense of distributions, i.e.,

$$\int_S \phi(z)\, d\mu(z) = \frac{1}{2\pi^2} \int_S K_\mu\, X\phi\, d\sigma(z) \tag{1.7}$$

for all smooth ϕ, provided that μ satisfies the necessary condition that $\int_S P\, d\mu = 0$ for all polynomials P. Note that (1.7) implies that K_μ is a CR function (in the sense of distributions) off the support of μ.

In attempting to use and understand Henkin's construction in the study of rational approximation on subsets of S, we were led to the analogue of the Cauchy-Green formula (1.1) that we present below. It plays the same role with respect to Henkin's formula (1.6) as the classical Cauchy-Green formula on the plane does to equation (1.4). The resulting formula, which is contained in our Theorems 2.1 and 3.1 below, is not new. It is given in a more general setting in Chen and Shaw ([**3**], see the remarks following Corollary 11.3.5) as a consequence of the theory of Henkin for solving the $\bar{\partial}_b$ equation on the boundary of a strictly convex domain in $\mathbf{C}^n$. Our approach to establishing this Cauchy-Green formula on the sphere in $\mathbf{C}^2$ is direct and elementary, and leads immediately to the property (1.6) of Henkin's transform K_μ.

Let $A(\mathbf{B})$ denote the algebra of functions holomorphic in the open unit ball $\mathbf{B}$ of $\mathbf{C}^2$ and continuous on its closure. We seek a kernel $H(\zeta, z)$, defined for $(\zeta, z) \in S \times S$, such that for all $\phi \in C^1(S)$, there exists $\Phi \in A(\mathbf{B})$ with

$$\phi(z) = \Phi(z) + c \int_S H(\zeta, z)\, \overline{\partial}\phi(\zeta) \wedge \omega(\zeta) \tag{1.8}$$

for all $z \in S$, where $\omega(\zeta) = d\zeta_1 \wedge d\zeta_2$, $\bar{\partial}\phi = (\partial\phi/\partial\bar{z}_1)d\bar{z}_1 + (\partial\phi/\partial\bar{z}_2)d\bar{z}_2$, and c is a universal constant. We call (1.8) a "Cauchy-Green formula for S". We will demand that H have the following properties:

a: $H(\zeta,z)$ is continuous on $S\times S\setminus\{z=\zeta\}$;
b: For all unitary transformations U of determinant 1, $H(U\zeta,Uz)=H(\zeta,z)$;
c: $\int_S |H(\zeta,e_1)|\ d\sigma(\zeta)<\infty$, where $e_1=(1,0)$, and $d\sigma$ is three-dimensional Hausdorff measure[1] on S.

Properties (b) and (c) together with the unitary invariance of $d\sigma$ imply that H is uniformly summable with respect to $d\sigma$, i.e., there exists a constant C so that

$$\int_S |H(\zeta,z)|\ d\sigma(\zeta)\le C,\ \ \forall z\in S \tag{1.9}$$

They also imply that the integral

$$K(z)\equiv\int_S H(\zeta,z)\ \overline{\partial}\phi(\zeta)\wedge\omega(\zeta) \tag{1.10}$$

appearing in (1.8) is finite for all $z\in S$, since $\overline{\partial}\phi\wedge\omega$ is absolutely continuous with respect to $d\sigma$. A routine calculation gives

$$\overline{\partial}\phi\wedge\omega=2(X\phi)\ d\sigma \tag{1.11}$$

on S, where X is the operator in (1.5), for smooth ϕ. We can say more about K:

LEMMA 1.1. *If H satisfies properties (a), (b) and (c), then K is continuous on S.*

PROOF. Fix $z\in S$. For $\epsilon>0$, put $S_\epsilon(z)=S\setminus\{|z-\zeta|\le\epsilon\}$ and $S'_\epsilon=S\cap\{|z-\zeta|\le\epsilon\}$. Let

$$K_\epsilon(z)=\int_{S_\epsilon(z)} H(\zeta,z)\ \overline{\partial}\phi(\zeta)\wedge\omega(\zeta)$$

Then K_ϵ is continuous on S, by property (a) of H. For all $z\in S$, by (1.11),

$$|K(z)-K_\epsilon(z)|=\left|\int_{S'_\epsilon(z)} H(\zeta,z)\ \overline{\partial}\phi(\zeta)\wedge\omega(\zeta)\right|\le M\int_{S'_\epsilon(z)}|H(\zeta,z)|d\sigma(\zeta)$$

where M is a constant independent of z and ϵ. Let $e_1=(1,0)$ and choose a unitary transformation U of $\mathbf{C}^2$ with $Ue_1=z$; then $U(S'_\epsilon(e_1))=S'_\epsilon(z)$. Then using property (b),

$$\int_{S'_\epsilon(z)}|H(\zeta,z)|\ d\sigma(\zeta)=\int_{S'_\epsilon(e_1)}|H(U\eta,Ue_1)|\ d\sigma(U\eta)=\int_{S'_\epsilon(e_1)}|H(\eta,e_1)|\ d\sigma(\eta)$$

Since $\int_S|H(\eta,e_1)|d\sigma(\eta)$ is finite by assumption (c),

$$\lim_{\epsilon\to 0}\int_{S'_\epsilon(e_1)}|H(\eta,e_1)|d\sigma(\eta)=0$$

It follows that $K_\epsilon\to K$ uniformly on S, and so K is continuous, as claimed. □

We say that a measure μ on S is orthogonal to polynomials if

$$\int_S P d\mu=0,\ \ \forall \text{ holomorphic polynomials } P \tag{1.12}$$

Given any measure μ on S, define

$$K_\mu(\zeta)=\int_S H(\zeta,z)\ d\mu(z),\ \ \zeta\in S \tag{1.13}$$

[1] $d\sigma$ is *not* normalized; $\sigma(S)=2\pi^2$.

LEMMA 1.2. *A kernel $H(\zeta, z)$ satisfying (a), (b) and (c) satisfies (1.8) if and only if for each measure μ on S orthogonal to polynomials*

$$\int_S \phi \, d\mu = c \int_S K_\mu \, \overline{\partial}\phi \wedge \omega \tag{1.14}$$

for all $\phi \in C^1(S)$.

PROOF. Suppose first that $H(\zeta, z)$ satisfies (a), (b), (c) and (1.8). Let μ be a measure on S orthogonal to polynomials. Fix $\phi \in C^1(S)$, and let $\Phi \in A(\mathbf{B})$ be as in (1.8). Since polynomials are dense in $A(\mathbf{B})$, $\int_S \Phi \, d\mu = 0$. Hence by (1.8),

$$\begin{aligned}
\int_S \phi(z) d\mu(z) &= \int_S \left(c \int_S H(\zeta, z) \, \overline{\partial}\phi(\zeta) \wedge \omega(\zeta) \right) d\mu(z) \\
&= \int_S \left(c \int_S H(\zeta, z) d\mu(z) \right) \overline{\partial}\phi(\zeta) \wedge \omega(\zeta) \\
&= c \int_S K_\mu(\zeta) \, \overline{\partial}\phi(\zeta) \wedge \omega(\zeta)
\end{aligned}$$

so that (1.14) holds. The application of Fubini's theorem is justified by (1.9).

Next, suppose that (1.14) holds, for H satisfying (a), (b) and (c). Choose a measure μ on S orthogonal to polynomials. Fix a function $\phi \in C^1(S)$, and define

$$\Phi(z) = \phi(z) - c \int_S H(\zeta, z) \, \overline{\partial}\phi(\zeta) \wedge \omega(\zeta)$$

By Lemma 1.1, Φ is continuous on S, and

$$\begin{aligned}
\int_S \Phi(z) d\mu(z) &= \int_S \phi(z) d\mu(z) - c \int_S \left(\int_S H(\zeta, z) d\mu(z) \right) \overline{\partial}\phi \wedge \omega(\zeta) \\
&= \int_S \phi(z) d\mu(z) - c \int_S K_\mu(\zeta) \, \overline{\partial}\phi(\zeta) \wedge \omega(\zeta) \\
&= 0
\end{aligned}$$

by (1.14). Since this holds for all μ orthogonal to polynomials, $\Phi \in A(\mathbf{B})$, and so (1.8) follows. ☐

In 1977, in [**4**] G. Henkin introduced the kernel

$$H(\zeta, z) = \frac{\bar{\zeta_1} \bar{z}_2 - \bar{\zeta_2} \bar{z}_1}{|1 - \langle z, \zeta \rangle|^2}, \quad \zeta, z \in S \tag{1.15}$$

where $\langle , \rangle$ denotes the Hermitian inner product $\langle z, \zeta \rangle = z_1 \bar{\zeta_1} + z_2 \bar{\zeta_2}$, and proved the formula (1.14) using this kernel. It is easy to check that H satisfies properties (a), (b) and (c) above. Formula (1.14) on S is actually very special case of a class of general integral formulae on smooth convex domains established in [**4**]. In her thesis [**5**], H.P. Lee gave an elementary proof of Henkin's formula for S; the paper [**8**] of Varopoulous also contains an exposition of Henkin's results on the sphere. For applications of Henkin's formula to rational approximation, see the paper [**6**] of Lee and Wermer.

In this paper, we shall

1. give a direct proof of (1.8), using Henkin's kernel (1.15);
2. give a formula for Φ, in terms of ϕ;
3. deduce an approximation result (Theorem 4.1) from (1.8).

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2. A Cauchy-Green Formula using Henkin's Kernel

With H as in (1.15) and $\phi \in C^1(S)$ as in section 1 put

$$K(z) = \int_S H(\zeta, z)\, \overline{\partial}\phi(\zeta) \wedge \omega(\zeta)$$

For $a \in \operatorname{int}(\triangle)$, put $r = \sqrt{1 - |a|^2}$ and denote by γ_a the circle $z_2 = r\tau$, $|\tau| = 1$ in the z_2-plane.

Lemma 2.1. *Fix $a \in \triangle$. For $n = 0, 1, 2, \ldots$ we have, putting $z = (a, z_2)$,*

$$\int_{\gamma_a} K(z) z_2^n \, dz_2 = 4\pi^2 \int_{\gamma_a} \phi(z) z_2^n \, dz_2 \tag{2.1}$$

Proof.

$$\begin{aligned}
\int_{\gamma_a} K(a, z_2) z_2^n \, dz_2 &= \int_{\gamma_a} \left(\int_S \frac{\bar{\zeta}_1 \bar{z}_2 - \bar{\zeta}_2 \bar{a}}{|1 - a\bar{\zeta}_1 - z_2\bar{\zeta}_2|^2} \, \overline{\partial}\phi(\zeta) \wedge \omega(\zeta) \right) z_2^n \, dz_2 \\
&= \int_S \left(\int_{|\tau|=1} \frac{(\bar{\zeta}_1 r\bar{\tau} - \bar{\zeta}_2 \bar{a}) r^{n+1} \tau^n \, d\tau}{(1 - a\bar{\zeta}_1 - r\tau\bar{\zeta}_2)(1 - \bar{a}\zeta_1 - r\bar{\tau}\zeta_2)} \right) \overline{\partial}\phi(\zeta) \wedge \omega(\zeta)
\end{aligned}$$

We denote the inner integral by $I(\zeta)$. Multiplying both numerator and denominator of the integrand by τ, we get

$$I(\zeta) = \int_{|\tau|=1} \frac{(\bar{\zeta}_1 r - \bar{\zeta}_2 \bar{a}\tau) r^{n+1} \tau^n \, d\tau}{r\bar{\zeta}_2 \left[\frac{1 - a\bar{\zeta}_1}{r\bar{\zeta}_2} - \tau \right] (1 - \bar{a}\zeta_1) \left[\tau - \frac{r\zeta_2}{1 - \bar{a}\zeta_1} \right]}$$

Let

$$\tau_1 = \frac{1 - a\bar{\zeta}_1}{r\bar{\zeta}_2}, \quad \text{and } \tau_2 = \frac{r\zeta_2}{1 - \bar{a}\zeta_1}$$

Note that $\tau_1 \overline{\tau_2} = 1$. We have

$$\begin{aligned}
|r\zeta_2|^2 - |1 - \bar{a}\zeta_1|^2 &= (1 - |a|^2)(1 - |\zeta_1|^2) - |1 - \bar{a}\zeta_1|^2 \\
&= 1 - |a|^2 - |\zeta_1|^2 + |a|^2||\zeta_1|^2 - 1 - |\bar{a}|^2|\zeta_1|^2 + \bar{a}\zeta_1 + a\bar{\zeta}_1 \\
&= -(|a|^2 + |\zeta_1|^2 - \bar{a}\zeta_1 - a\bar{\zeta}_1) \\
&= -|a - \zeta_1|^2
\end{aligned}$$

It follows that $|r\zeta_2|^2 < |1-\bar{a}\zeta_1|^2$ unless $a = \zeta_1$, and so $|\tau_2| < 1$, and $|\tau_1| > 1$, for $\zeta_1 \neq a$. The Residue Theorem gives

$$\begin{aligned}
\frac{1}{2\pi i} I(\zeta) &= \left[\frac{\bar{\zeta}_1 r - \bar{\zeta}_2 \bar{a}\tau_2}{r\bar{\zeta}_2(\frac{1-a\bar{\zeta}_1}{r\bar{\zeta}_2} - \tau_2)(1-\bar{a}\zeta_1)}\right] r^{n+1}\tau_2^n \\
&= \left[\frac{\bar{\zeta}_1 r - \bar{\zeta}_2 \bar{a}(\frac{r\zeta_2}{1-\bar{a}\zeta_1})}{r\bar{\zeta}_2(\frac{1-a\bar{\zeta}_1}{r\bar{\zeta}_2} - \frac{r\zeta_2}{1-\bar{a}\zeta_1})(1-\bar{a}\zeta_1)}\right] r^{n+1}\tau_2^n \\
&= \left[\frac{(1-\bar{a}\zeta_1)\bar{\zeta}_1 r - (\bar{\zeta}_2\bar{a}) r\zeta_2}{r\bar{\zeta}_2(\frac{|1-\bar{a}\zeta_1|^2}{r\bar{\zeta}_2} - r\zeta_2)(1-\bar{a}\zeta_1)}\right] r^{n+1}\tau_2^n \\
&= \left[\frac{\bar{\zeta}_1 r - \bar{a}|\zeta_1|^2 r - \bar{a}|\zeta_2|^2 r}{|1-\bar{a}\zeta_1|^2 - r^2|\zeta_2|^2}\right] \frac{r^{n+1}\tau_2^n}{1-\bar{a}\zeta_1} \\
&= \left[\frac{(\bar{\zeta}_1 - \bar{a})r}{|\zeta_1 - a|^2}\right] \frac{r^{n+1}\tau_2^n}{1-\bar{a}\zeta_1} \\
&= \left(\frac{r}{\zeta_1 - a}\right) r^{n+1} \left(\frac{r\zeta_2}{1-\bar{a}\zeta_1}\right)^n \left(\frac{1}{1-\bar{a}\zeta_1}\right) \\
&= \frac{r^{2n+2}\zeta_2^n}{(1-\bar{a}\zeta_1)^{n+1}} \left(\frac{1}{\zeta_1 - a}\right)
\end{aligned}$$

Thus

$$I(\zeta) = 2\pi i \cdot \frac{r^{2n+2}\zeta_2^n}{(1-\bar{a}\zeta_1)^{n+1}} \left(\frac{1}{\zeta_1 - a}\right) \tag{2.2}$$

Let S_ϵ be the part of S lying over the region

$$\{|\zeta_1 - a| \geq \epsilon\} \cap \{|\zeta_1| \leq 1\}$$

in the ζ_1-plane. Let T_ϵ denote the boundary of S_ϵ. We claim that

$$\int_{\gamma_a} K(z) z_2^n \, dz_2 = -\lim_{\epsilon \to 0} \left[\int_{T_\epsilon} \phi(\zeta) I(\zeta) \omega(\zeta)\right] \tag{2.3}$$

To establish the claim, note that

$$\begin{aligned}
\int_{\gamma_a} K(z) z_2^n \, dz_2 &= \int_S \overline{\partial}\phi \wedge \omega \cdot I \\
&= \lim_{\epsilon \to 0} \int_{S_\epsilon} \overline{\partial}\phi \wedge \omega \cdot I \\
&= \lim_{\epsilon \to 0} \int_{S_\epsilon} d(\phi \, \omega I)
\end{aligned}$$

since I is holomorphic on S_ϵ for $\epsilon > 0$. By Stokes' Theorem, the latter integral equals

$$-\int_{T_\epsilon} \phi \, \omega I$$

proving the claim.

Note that T_ϵ is the torus

$$\zeta_1 = a + \epsilon e^{i\theta}, \;\; \zeta_2 = \sqrt{1 - |\zeta_1|^2} e^{i\psi}, \;\; 0 \leq \theta, \psi \leq 2\pi.$$

On T_ϵ we have the following relations:

$$\phi(\zeta) = \phi(a, re^{i\psi}) + O(\epsilon);$$

$$\frac{\zeta_2^n}{(1-\bar{a}\zeta_1)^{n+1}} = \frac{r^n e^{in\psi}}{(1-|a|^2)^{n+1}} + O(\epsilon) = \frac{r^n e^{in\psi}}{r^{2n+2}} + O(\epsilon);$$

$$d\zeta_1 = i\epsilon e^{i\theta} d\theta, \ \ d\bar{\zeta_1} = -i\epsilon e^{-i\theta} d\theta;$$

$$d\zeta_2 = \frac{-\zeta_1 d\bar{\zeta_1} - \bar{\zeta_1} d\zeta_1}{2\sqrt{1-|\zeta_1|^2}} e^{i\psi} + i\sqrt{1-|\zeta_1|^2} e^{i\psi} d\psi = ire^{i\psi} d\psi + O(\epsilon);$$

$$\frac{1}{\zeta_1 - a} = \frac{1}{\epsilon e^{i\theta}}.$$

Using this information together with (2.2) and (2.3) we obtain

$$\int_{\gamma_a} K(z) z_2^n = \lim_{\epsilon \to 0} \left[-2\pi i \int_{T_\epsilon} \phi(\zeta) r^{2n+2} \frac{\zeta_2^n}{(1-\bar{a}\zeta_1)^{n+1}} \frac{1}{(\zeta_1 - a)} \, d\zeta_1 \wedge d\zeta_2 \right]$$

For fixed ϵ, we rewrite the expression in brackets as

$$-2\pi i \int_{T_\epsilon} \phi(a, re^{i\psi}) r^n e^{in\psi} i d\theta \wedge ire^{i\psi} d\psi + O(\epsilon)$$

Letting ϵ go to zero we obtain

$$\begin{aligned} \int_{\gamma_a} K(z) z_2^n \, dz_2 &= 2\pi \int_0^{2\pi} d\theta \int_0^{2\pi} \phi(a, re^{i\psi}) (re^{i\psi})^n (ire^{i\psi}) \, d\psi \\ &= 4\pi^2 \int_{\gamma_a} \phi(z) z_2^n \, dz_2 \end{aligned}$$

This completes the proof of (2.1) and Lemma 2.1. □

Next, we define an operator T on $C^1(S)$ as follows:

(2.4) $$(T\phi)(z) = 4\pi^2 \phi(z) - K(z), \quad \text{for } z \in S, \phi \in C^1(S)$$

Letting X denote the tangential Cauchy-Riemann operator on S as in section 1, using (1.11) we can write

$$T\phi = 4\pi^2 \phi - \int_S H(\zeta, z) \, (X\phi)(\zeta) d\sigma(\zeta)$$

LEMMA 2.2. *Fix $\phi \in C^1(S)$. Let L be a complex line in $\mathbf{C}^2$. Then the restriction of $T(\phi)$ to $L \cap S$ extends analytically to $L \cap \mathbf{B}$.*

PROOF. Lemma 2.1 gives us, for each $a \in \text{int}(\triangle)$, that

(2.5) $$\int_{\gamma_a} (T\phi)(z) z_2^n \, dz_2 = 0, \ n = 0, 1, 2, \ldots$$

Note that $\gamma_a = L_a \cap S$, where L_a is the line $\{z_1 = a\}$. Then (2.5) implies that $T\phi$ extends analytically to the disk $L_a \cap \mathbf{B}$. Using the unitary invariance of $H, \sigma,$ and X, it is not hard to check that for all $\phi \in C^1(S)$,

(2.6) $$(T\phi) \circ U = T(\phi \circ U)$$

Fix a complex line L. Let N denote the complex line passing through the origin which is orthogonal to L, and let z^0 denote the intersection point $N \cap L$. Write $L = \{z^0 + \zeta t \mid t \in \mathbf{C}\}$ for some unit vector ζ. If U is a unitary transformation with $Ue_2 = \zeta$, where $e_2 = (0,1)$ then U maps the line $\{z_2 = 0\}$ to N, and maps some point $(a, 0)$ to z^0. Then $U((a,0) + t(0,1)) = z^0 + t\zeta$, for all $t \in \mathbf{C}$. So U maps the line L_a to L and maps the disk $L_a \cap \mathbf{B}$ to $L \cap \mathbf{B}$. By (2.6), $T\phi\,|_{L\cap S}$ extends analytically to the disk $L\cap\mathbf{B}$ if and only if $(T\phi)\circ U\,|_{L_a\cap S}$ extends to $L_a \cap \mathbf{B}$. This last is true by (2.5), as we have noted earlier, and so the proof is complete. □

By Lemma 1.1, since H satisfies properties (a), (b) and (c) of section 1, K and thus $T\phi$ are continuous on S. By Lemma 2.2, $T\phi$ has the "one-dimensional extension property" as defined by Stout in [**7**], p. 105. A theorem of Agranovskii and Val'skii [**1**] then gives that $T\phi$ lies in the ball algebra $A(\mathbf{B})$. Putting $\Phi = T(\phi)$, we have arrived at

THEOREM 2.3. *Let $\phi \in C^1(S)$. Then there exists $\Phi \in A(\mathbf{B})$ such that*

$$4\pi^2\phi(z) = \Phi(z) + \int_S H(\zeta,z)\,\overline{\partial}\phi(\zeta)\wedge\omega(\zeta)$$

where H is Henkin's kernel

$$H(\zeta,z) = \frac{\bar{\zeta}_1\bar{z}_2 - \bar{\zeta}_2\bar{z}_1}{|1-<z,\zeta>|^2}$$

3. The Cauchy-Green formula and the Cauchy transform

In this section we identify the ball algebra function Φ appearing in Theorem 2.3 as a certain principal value of the Cauchy transform of ϕ. The Cauchy kernel for $\mathbf{B}$ is

$$C(z,\zeta) = \frac{1}{(1-<z,\zeta>)^2}$$

For $z \in S$ we set

$$N_\epsilon(z) = \{\zeta \in S : |<\zeta,z>| > 1-\epsilon\}$$

and we denote the boundary of $N_\epsilon(z)$ by $\Gamma_\epsilon(z)$.

THEOREM 3.1. *Fix $\phi \in C^1(S)$. If Φ is as in Theorem 2.3, then for $z \in S$,*

$$\Phi(z) = 2\lim_{\epsilon\to 0}\int_{S\setminus N_\epsilon(z)} \phi(\zeta)C(z,\zeta)\,d\sigma(\zeta)$$

REMARK 3.2. Since $C(z,\cdot) \notin L^1(d\sigma)$, it is not immediate that the limit in Theorem 3.1 exists.

PROOF. As in sections 1 and 2, set

$$K(z) = \int_S H(\zeta,z)\,\overline{\partial}\phi(\zeta)\wedge\omega(\zeta) = \lim_{\epsilon\to 0}\int_{S\setminus N_\epsilon(z)} H(\zeta,z)\,\overline{\partial}\phi(\zeta)\wedge\omega(\zeta)$$

For $\epsilon > 0$ fixed,

$$\begin{aligned}\int_{S\setminus N_\epsilon(z)} H(\zeta,z)\,\overline{\partial}\phi(\zeta)\wedge\omega(\zeta) &= \int_{S\setminus N_\epsilon(z)} d[H(\zeta,z)\phi(\zeta)\wedge\omega(\zeta)] \\ &\quad -\int_{S\setminus N_\epsilon(z)} [\overline{\partial}H(\zeta,z)]\wedge\phi(\zeta)\wedge\omega(\zeta) \\ &= \int_{\Gamma_\epsilon(z)} H(\zeta,z)\,\overline{\partial}\phi(\zeta)\wedge\omega(\zeta) \\ &\quad -2\int_{S\setminus N_\epsilon(z)} (XH)(\zeta,z)\,\phi(\zeta)\,d\sigma(\zeta)\end{aligned}$$

by Stokes' theorem, if $\Gamma_\epsilon(z)$ is oriented as the boundary of $S \setminus N_\epsilon(z)$. We have also used equation (1.11) from section 1. A computation shows (differentiation is in the ζ variable)

$$(XH)(\zeta, z) = -C(z, \zeta)$$

so that

$$K(z) = \lim_{\epsilon \to 0} \left[\int_{\Gamma_\epsilon(z)} H(\zeta, z)\ \phi(\zeta) \wedge \omega(\zeta) - 2 \int_{S \setminus N_\epsilon(z)} C(\zeta, z)\ \phi(\zeta) d\sigma(\zeta) \right]$$

Since

$$\Phi(z) = 4\pi^2 \phi(z) - K(z)$$

by Theorem 2.3, the proof will be complete if we can show that

$$\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon(z)} H(\zeta, z)\ \phi(\zeta) \wedge \omega(\zeta) = 4\pi^2 \phi(z) \tag{3.1}$$

To establish (3.1), choose a unitary map U with $Ue_1 = z$. Then for fixed $\epsilon > 0$,

$$\int_{\Gamma_\epsilon(z)} H(\zeta, z)\ \phi(\zeta) \wedge \omega(\zeta) = \int_{\Gamma_\epsilon(e_1)} H(\eta, e_1)\ (\phi \circ U)(\eta) \wedge \omega(\eta)$$

The torus $\Gamma_\epsilon(e_1) = \{\eta : |\eta_1| = 1 - \epsilon\}$, oriented as the boundary of $S \setminus N_\epsilon(e_1)$, is parametrized by

$$\eta_1 = (1 - \epsilon) e^{i\theta_1},\ \eta_2 = r_\epsilon e^{i\theta_2},\ \ 0 \le \theta_1, \theta_2 \le 2\pi$$

where

$$r_\epsilon = \sqrt{1 - (1 - \epsilon)^2}$$

Then on $\Gamma_\epsilon(e_1)$,

$$\omega(\eta) = d\eta_1 \wedge d\eta_2 = -(1 - \epsilon) r_\epsilon e^{i\theta_1} e^{i\theta_2} d\theta_1 \wedge d\theta_2,$$

$$(\phi \circ U)(\eta) = (\phi \circ U)(e^{i\theta_1}, 0) + O(\epsilon),$$

and

$$H(\eta, e_1) = \frac{-r_\epsilon e^{-i\theta_2}}{|1 - (1 - \epsilon) e^{i\theta_1}|^2}$$

which gives

$$\begin{aligned} \int_{\Gamma_\epsilon(z)} H(\zeta, z)\ \phi(\zeta) \wedge \omega(\zeta) &= \int_{\Gamma_\epsilon(e_1)} H(\eta, e_1)\ (\phi \circ U)(\eta) \wedge \omega(\eta) \\ &= \int_0^{2\pi} \int_0^{2\pi} \frac{(1 - \epsilon) r_\epsilon^2 e^{i\theta_1} (\phi \circ U)(e^{i\theta_1}, 0)}{|1 - (1 - \epsilon) e^{i\theta_1}|^2} d\theta_1 d\theta_2 + I_\epsilon \end{aligned} \tag{3.2}$$

where

$$|I_\epsilon| \le C \int_0^{2\pi} \int_0^{2\pi} \frac{r_\epsilon^2}{|1 - (1 - \epsilon) e^{i\theta_1}|^2} d\theta_1 d\theta_2$$

for some $C > 0$. An application of the Poisson integral formula shows that the first integral in (3.2) converges to $4\pi^2 (\phi \circ U)(e_1) = 4\pi^2 \phi(z)$ as $\epsilon \to 0$, while $\lim_{\epsilon \to 0} I_\epsilon = 0$. This completes the proof. □

4. An Approximation Theorem

Fix $\phi \in C^1(S)$. The quantity

$$\mathrm{dist}(\phi, A(\mathbf{B})) = \inf\{\|\phi - g\| : g \in A(\mathbf{B})\}$$

where $\|\cdot\|$ is the uniform norm on S measures how closely ϕ can be approximated by polynomials on S.

THEOREM 4.1. *There exists $C > 0$ so that for all $\phi \in C^1(S)$,*

$$dist(\phi, A(\mathbf{B})) \leq C\|X\phi\|$$

PROOF. Let $\|H\|_1$ denote the $L^1 - d\sigma$ norm of Henkin's kernel $H(\cdot, z)$ (which is independent of $z \in S$). By the representation in Theorem 2.3, there exists $\Phi \in A(\mathbf{B})$ so that for $z \in S$,

$$\begin{aligned} |4\pi^2\phi(z) - \Phi(z)| &= \left|\int_S H(\zeta, z)\, \overline{\partial}\phi(\zeta) \wedge \omega(\zeta)\right| \\ &= 2\left|\int_S H(\zeta, z)(X\phi)(\zeta)d\sigma(\zeta)\right| \\ &\leq 2\|H\|_1\|X\phi\| \end{aligned}$$

from which the result follows. □

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