GLOBAL MAPPING PROPERTIES OF RATIONAL FUNCTIONS

Cristina Ballantine[1] and Dorin Ghisa[2]

Abstract

We investigate the fundamental domains of rational functions and provide visualizations for relevant examples. The fundamental domains give a thorough understanding of the global properties of the functions studied.

1 Introduction

Any rational function $f(z)$ can be viewed as the canonical projection of a branched covering Riemann surface $(\widehat{\mathbb{C}}, f)$ of the Riemann sphere $\widehat{\mathbb{C}}$. Indeed, f is locally injective in the neighborhood of every point $z \in \widehat{\mathbb{C}}$, except for the points z_k, which are solutions of the equation $f'(z) = 0$ and the points c_j which are multiple poles of f. In [Bar-G] we have studied global mapping properties of Blaschke products, showing that every Blaschke product $w = B(z)$ of degree n induces partitions of $\widehat{\mathbb{C}}$ into n sets whose interior is mapped conformally by B onto $\widehat{\mathbb{C}} \setminus L$, where L is a cut. Following [A, p. 98] we called these sets *fundamental regions* or *domains*.

The fundamental regions have played an important role in the theory of automorphic functions. In fact, a fundamental region of a group of transformations is a fundamental region of an automorphic function with respect to that group. These regions characterize the global mapping properties of automorphic functions. In this paper we show that any rational function f has similar properties. Moreover, once the fundamental regions of f are known, invariants of f can be found, *i.e.* mappings U_k of the Riemann sphere on itself such that, for every $z \in \widehat{\mathbb{C}}$, we have $f \circ U_k(z) = f(z)$. Obviously, the set of these invariants is a cyclic group of order n. They are the cover transformations (see [A-S, p. 37]) of $(\widehat{\mathbb{C}}, f)$ and we can extend the concept of automorphic function to such a group. Using this terminology, the main result of this paper shows that any rational function f is an automorphic function with respect to the group of cover transformations of $(\widehat{\mathbb{C}}, f)$. The proof is constructive and we use the technique of simultaneous continuations developed in [Bar-G] in order to find fundamental regions for f.

To visualize the fundamental regions, we color a set of annuli centered at the origin of the w-plane in different colors with saturation increasing counter-clockwise (*i.e.*, determined by the argument of each point) and brightness increasing outward (*i.e.*, determined by the absolute value each point) and impose the same color, saturation and brightness to the pre-image of every point in these annuli.

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2 A Simple Example: Linear Fractional Transformations

We visualize the linear fractional transformation

$$w = f(z) = \frac{az+b}{cz+d}, \quad \text{where } ad - bc \neq 0$$

as follows.

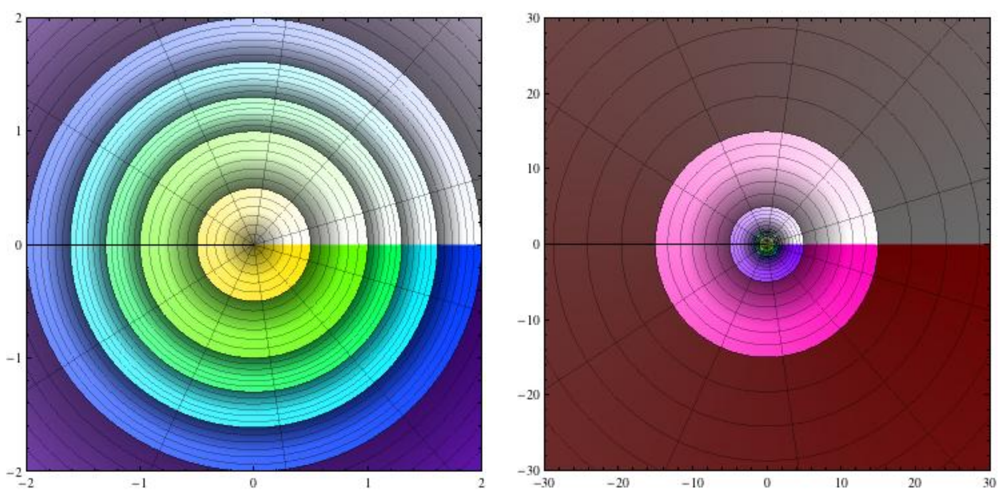
Consider the circle $w = re^{i\theta}$ (r is fixed). Its pre-image by f is $z(r,\theta) = \frac{-dw+b}{cw-a}$.

$$\lim_{r\to 0} z(r,\theta) = -\frac{b}{a}, \text{ and } \lim_{r\to\infty} z(r,\theta) = -\frac{d}{c}.$$

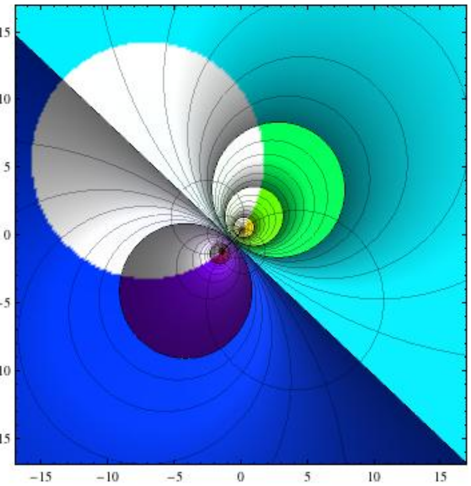
If r is small, the pre-image is a circle containing $-\frac{b}{a}$. If r is large, the pre-image is a circle containing $-\frac{d}{c}$. If $r = \left|\frac{a}{c}\right|$, the pre-image of the circle is a line (perpendicular to the line through $-\frac{b}{a}$ and $-\frac{d}{c}$, since f is conformal).

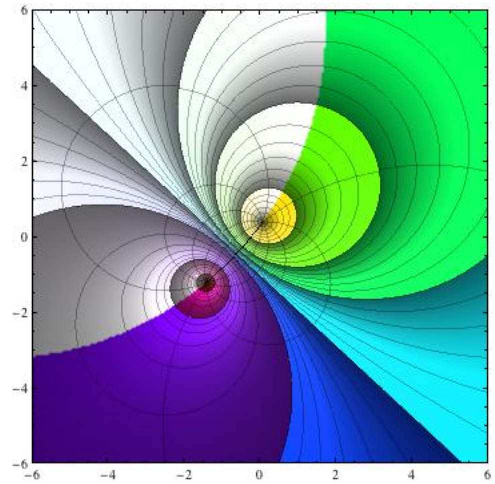
Example 1: $f(z) = \dfrac{(2+3i)z+(1-i)}{(1+2i)z+(-1+4i)}$.

The pre-images of the annuli



under the mapping f are shown below.





We notice that f has a single fundamental domain.

For reasons of clarity, in the images above we provide a zoom of the colored annuli and of the fundamental domains.

3 Mapping Properties of the Second Degree Rational Functions

A study of the second degree rational functions can be found in [N, p.266]. We use Nehari's results in order to illustrate some of the mapping properties of these functions. The main result found in [N] relevant to this topic is that any mapping $w = f(z) = \dfrac{a_1 z^2 + a_2 z + a_3}{b_1 z^2 + b_2 z + b_3}$ can be written under the form

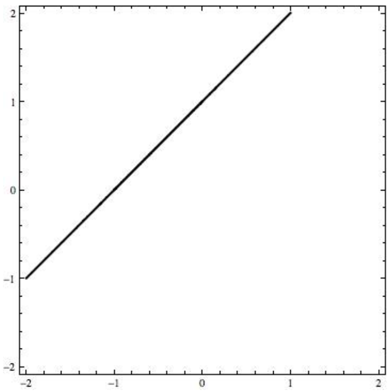
$$f(z) = S_2 \circ T \circ S_1, \tag{1}$$

where S_1 and S_2 are Möbius transformations and $\eta = T(\zeta) = \zeta^2$. Indeed, to prove this statement we only need to determine six essential parameters of the two unknown Möbius transformations S_1 and S_2 such that (1) is true, which is allays possible.

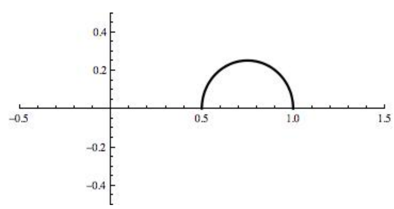
The function $\zeta = S_1(z)$ transforms the z-plane into the ζ-plane, such that a circle (see Example 3 below) or a line L (see Example 2 below) corresponds to the real axis from the ζ-plane. The function $\eta = T(\zeta) = \zeta^2$ transforms each one of the upper and the lower half-planes of the ζ-plane into the whole η-plane with a cut alongside the real half-axis. Finally, the function $w = S_2(\eta)$ transforms the η-plane into the w-plane and the real half-axis into an arc of a circle or a half line L'. Summing up, f maps conformally each one of the two domains determined by L onto the whole w-plane with a cut alongside L'. Thus, for such a function f, the fundamental domains can allays be taken the two domains mapped by S_1 onto the upper and the lower half planes.

Example 2: We illustrate the case where $f(z) = \dfrac{(1+i)z^2 + 4z + 1 - i}{(2+i)z^2 + 6z + 2 - i}$. In this case $S_1(z) =$

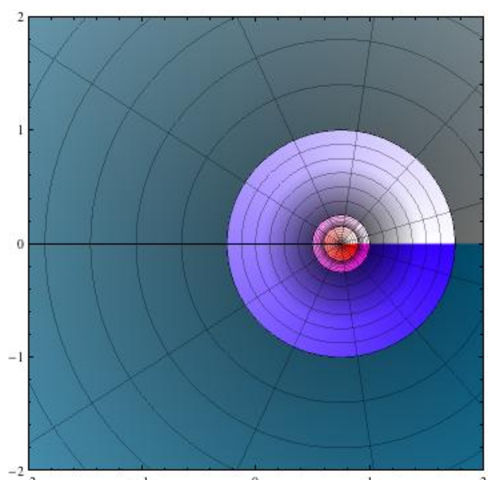
$\dfrac{z+1}{z-i}$ and $S_2(z) = \dfrac{z+i}{2z+i}$. The pre-image of the real axis under S_1 is the line $z = \dfrac{1+ti}{t-1}$ shown below.



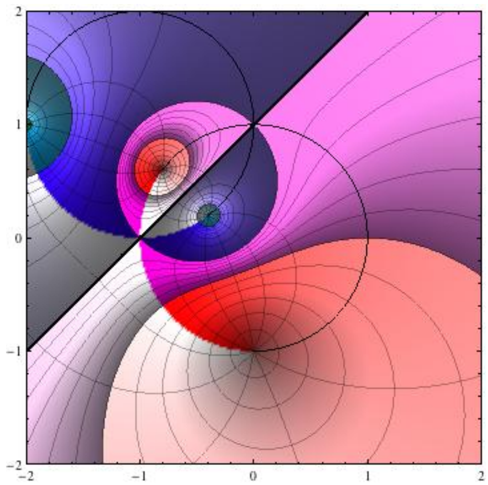
The image of the positive real half axis under S_2 is the semicircle of radius 0.25 centered at 0.75.



For the visualization, we consider colored annuli centered at $(0.75, 0)$.



We visualize the fundamental domains of f in Figure 1(a), by considering pre-images of these annuli under f.



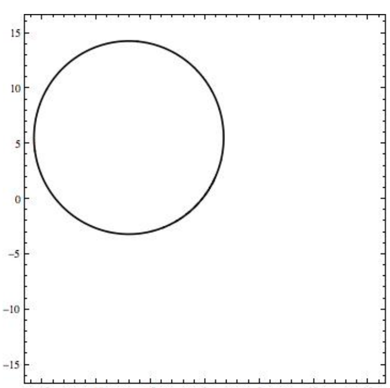
Note: f has two funadamental domains. They are precisely the regions delimited by the pre-image L of the real axis under S_1.

Example 3: We illustrate the case $f(z) = \dfrac{(8-17i)+(6-16i)z-(9-9i)z^2}{(8-19i)+(16-14i)z-(14-21i)z^2}$.

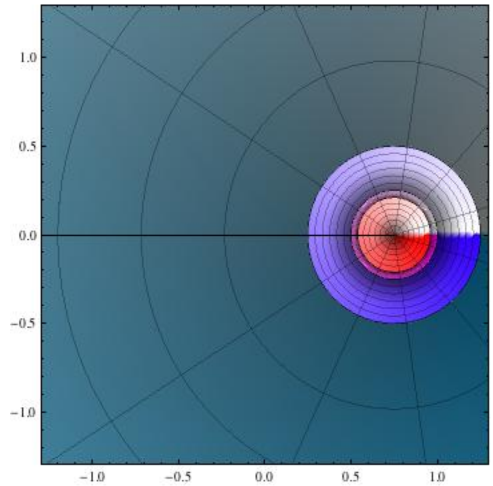
Then, S_2 is the same Möbius transformation as in the previous example and S_1 is the Möbius transformation illustrated in the previous section.

Thus, $S_1(z) = \dfrac{(2+3i)z+(1-i)}{(1+2i)z+(-1+4i)}$ and $S_2(z) = \dfrac{z+i}{2z+i}$.

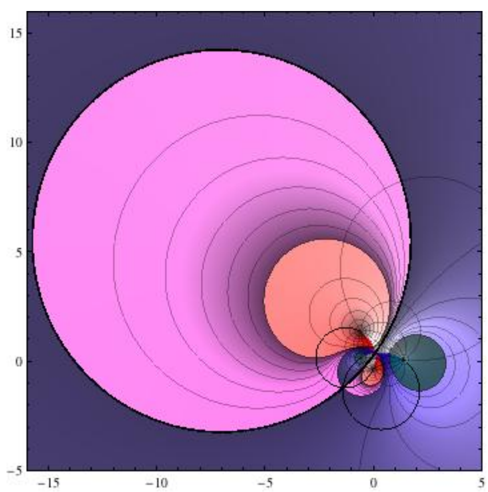
The pre-image of the real axis under S_1 is the *circle* shown below.

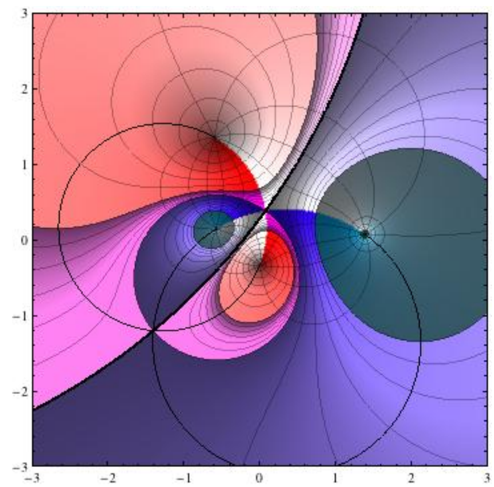


The pre-images of the annuli centered at 0.75



under the mapping f are:





Note: f has two fundamental domains: the interior of the disk whose boundary is the pre-image of the real axis under S_1 and the exterior of this disk.

4 Mapping properties of Blaschke Quotients

In [B-G] we studied the mapping properties of Blaschke quotients B of a special type, namely such that for every $z \in \widehat{\mathbb{C}}$, $B \circ h(z) = h \circ B(z)$, where $h(z) = -1/\overline{z}$. Such a rational function has the particularity that its poles and zeros appear in pairs which are opposite to each other and if z_k is a pole of order p of B, then $1/\overline{z}_k$ is a zero of order p of B and vice-versa. The point $z = 0$ is a zero or a pole of B of an odd order and therefore ∞ is a pole, respectively a zero, of the same order.

The main result of [B-G] shows that, for a Blaschke quotient of degree n of such a type, there is a partition of $\widehat{\mathbb{C}}$ in $2n$ simply connected sets such that the interior of each one of them is mapped conformally by B either on the open unit disc (i-set), or on the exterior of the closed unit disc (e-set). The interior of the union of an i-set and an adjacent e-set is mapped conformally by B on the Riemann sphere with a slit. The map is continuous (with respect to the spheric metric) on the borders, except for the branch points. Here, we prove that a similar property holds for any finite Blaschke quotient.

Let $B(z) = B_1(z)/B_2(z)$ be a Blaschke quotient of degree n, *i.e.* the quotient of two finite Blaschke products B_1 and B_2 of degrees n_1, respectively n_2, such that $\max\{n_1, n_2\} = n$. The function B is locally injective, except for the set of points $H_1 = \{b_1, b_2, ..., b_m\}$, which are solutions of the equation $B'(z) = 0$. Consequently, $(\widehat{\mathbb{C}}, B)$ is a branched covering Riemann surface of $\widehat{\mathbb{C}}$ having H_1 as set of branch points. In other words, $(\widehat{\mathbb{C}} \setminus H_1, B)$ is a smooth covering Riemann surface of $\widehat{\mathbb{C}}$.

Theorem 1 *For every Blaschke quotient B of degree n there is a partition of $\widehat{C}$ into n sets symmetric with respect to the unit circle whose interior Ω_k is mapped each one conformally by B on $\widehat{\mathbb{C}} \setminus L$, where L is a cut. Moreover, $B : \overline{\Omega}_k \to \widehat{C}$ is surjective.*

Proof: Let $H_2 = \{z_1, z_2, ..., z_n\}$ be the solutions of the equation $B(z) = e^{i\theta}$, where $\theta \in R$ has been chosen such that $H_1 \cap H_2 = \emptyset$. It is obvious that such a choice is always possible. Since the image of the unit circle by B is the unit circle, at least one of the points z_k belongs to the unit circle. Also, since $B(1/\overline{z}) = 1/\overline{B(z)}$, the solutions which are not on the unit circle, must be two by two symmetric with respect to the unit circle.

If we perform simultaneous continuation from every z_j over the unit circle (starting from $e^{i\theta}$), we obtain arcs $\gamma_{j,j'}$ starting at $z_j \in H_2$ and ending at some point $z_{j'} \in H_2$. Some of these arcs might cross each other, but this can happen only at the points in H_1 since these are the only points where the injectivity of $B(z)$ is violated.

Let $W = \{w_1, w_2, ..., w_p\}$, where $w_k = B(b_k)$, $|b_k| < 1, b_k \in H_1$ and w_k are not points of intersection of $\gamma_{j,j'}$. We connect $e^{i\theta}, w_1, ..., w_p$ by a polygonal line Γ with no self intersection and perform simultaneous continuation over Γ from all $z_j \in H_2$. The domains bounded by the pre-image of Γ and the arcs $\gamma_{j,j'}$ are mapped by B either on the unit disc (i-domains) or on the exterior of the unit circle (e-domains). Indeed, every one of these domains $\Omega_{j,j'}$ is bounded by an arc $\gamma_{j,j'}$ whose image by B is the unit circle, and by an arc having the end points in z_j and $z_{j'}$ whose image by B is a part of Γ. The previous affirmation follows from the conformal correspondence theorem (see [N, p. 154]). It is obvious that every i-domain has a symmetric e-domain with respect to the unit circle and vice-versa. An i-domain and an adjacent e-domain are always separated by an arc $\gamma_{j,j'}$ and their union to which the open $\gamma_{j,j'}$ is added as a point set constitutes a fundamental domain Ω_j of B. If we denote $L = \Gamma \cup \widetilde{\Gamma}$, where $\widetilde{\Gamma}$ is the symmetric of Γ with respect to the unit circle, then it is obvious that B maps conformally every Ω_k on $\widehat{\mathbb{C}} \setminus L$ and the mapping $B : \overline{\Omega}_k \to \widehat{\mathbb{C}}$ is surjective, which completely proves the theorem.

Example 4: Let $a_1 = \dfrac{1}{4}e^{\frac{\pi i}{6}}$, $a_2 = \dfrac{1}{3}e^{-\frac{\pi i}{5}}$ and $b = \dfrac{1}{2}e^{\frac{2\pi i}{3}}$.

$$B_1(z) = \left(\frac{\overline{a_1}}{|a_1|}\frac{z - a_1}{\overline{a_1}z - 1}\right)^2 \cdot \frac{\overline{a_2}}{|a_2|}\frac{z - a_2}{\overline{a_2}z - 1}$$

$$B_2(z) = \left(\frac{\overline{b}}{|b|}\frac{z-b}{\overline{b}z-1}\right)^2.$$

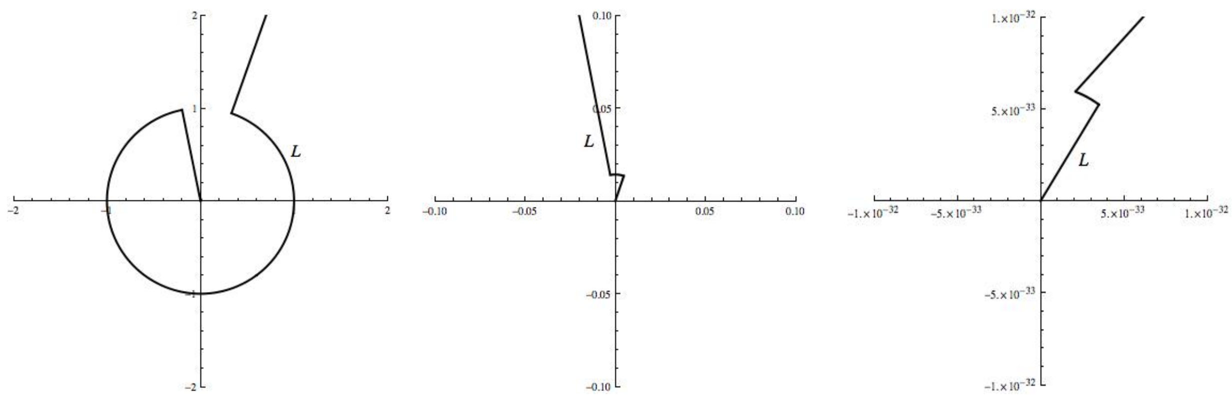
Then,

$$B(z) = \frac{B_1(z)}{B_2(z)} = \frac{e^{-\frac{4i\pi}{5}}\left(-\frac{1}{4}e^{\frac{i\pi}{6}}+z\right)^2\left(-\frac{1}{3}e^{-\frac{i\pi}{5}}+z\right)\left(-1+\frac{1}{2}e^{-\frac{2i\pi}{3}}z\right)^2}{\left(-\frac{1}{2}e^{\frac{2i\pi}{3}}+z\right)^2\left(-1+\frac{1}{4}e^{-\frac{i\pi}{6}}z)\right)^2\left(-1+\frac{1}{3}e^{\frac{i\pi}{5}}z\right)}$$

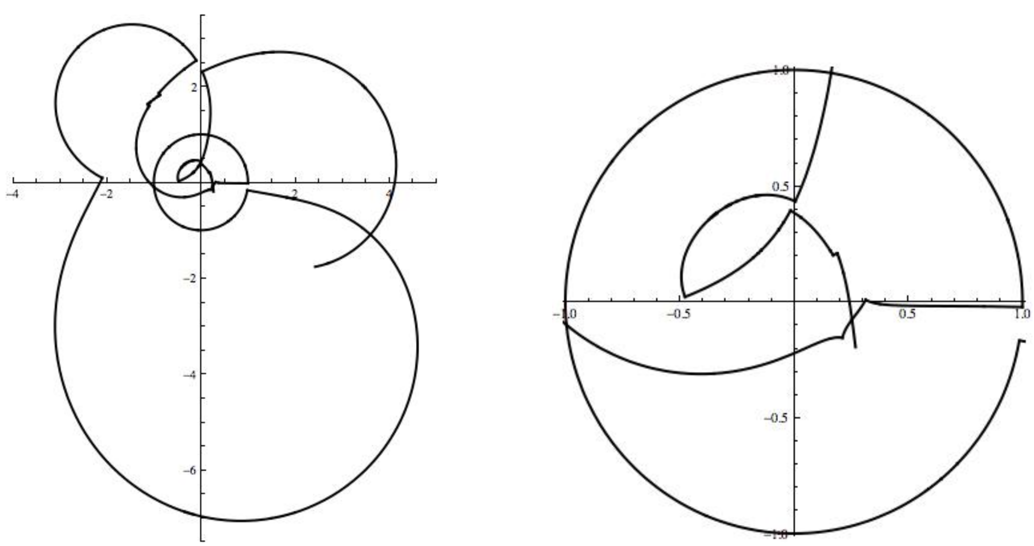
is a Blaschke quotient of degree 5.

$H_1 = \{0.216506+0.125i, -1.+1.73205i, 0.162638+0.986686i, 0.254261-0.0769968i, -0.994981-0.100059i, 3.6026-1.09096i\}$.

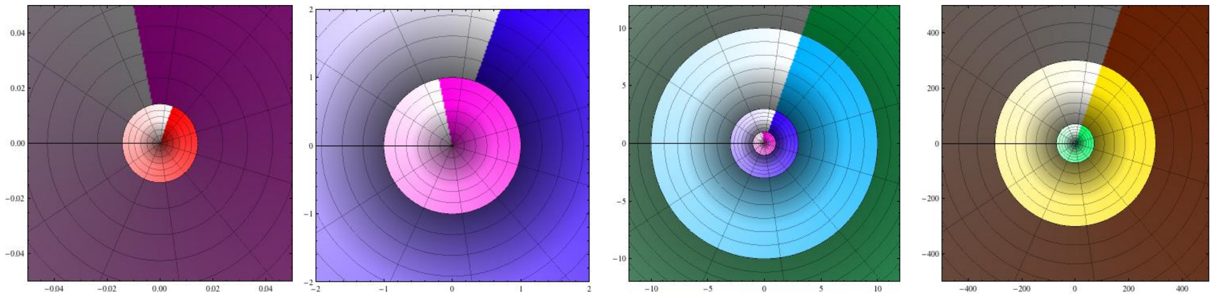
A polygonal line L passing though the images of the branch points is shown below.



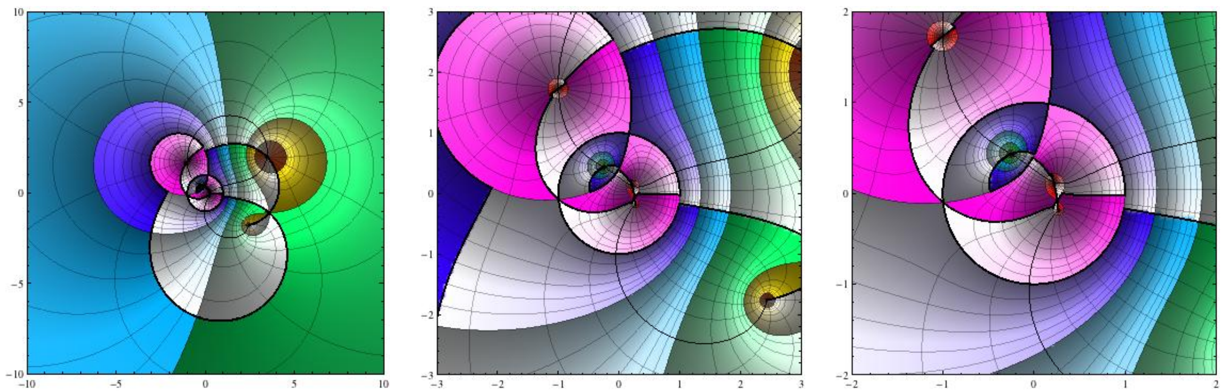
The pre-image of L under B is shown below.



We consider a collection of colored annuli.



The pre-image of these annuli under B are shown below.



For a better view, each figure above shows several zoomed images.
In the next section we show that a similar property is true for any rational function.

5 Mapping Properties of Arbitrary Rational Functions

Let $w = f(z)$ be a rational function with zeros $a_1, a_2, ..., a_p$ and poles $b_1, b_2, ..., b_q$. Let α_i be the multiplicity of a_i and β_j be the multiplicity of b_j. Then, the *degree* of f is $n = \max\{u, v\}$, where $u = \alpha_1 + \alpha_2 + ... + \alpha_p$ and $v = \beta_1 + \beta_2 + ... + \beta_q$.

If $\lim\limits_{z\to\infty} f(z) = 0$, $n = v$ and $a_0 = \infty$ is said to be a zero of multiplicity $\alpha_0 = n - u$ of f. If $\lim\limits_{z\to\infty} f(z) = \infty$, $n = u$ and $b_0 = \infty$ is said to be a pole of multiplicity $\beta_0 = n - v$ of f.

Theorem 2 *Every rational function f of degree n defines a partition of $\widehat{\mathbb{C}}$ into n sets whose interior is mapped conformally by f on $\widehat{\mathbb{C}} \setminus L$, where L is a cut. The mapping can be analytically extended to the boundaries, except for a number $\leq n$ of common points z_j of those boundaries in the neighborhood of which f is of the form*

(i) $f(z) = w_j + (z - z_j)^k h(z)$, when $f(z_j) = w_j$,

(ii) $f(z) = (z - z_j)^{-k} h(z)$, when $f(z_j) = \infty$,

(iii) $f(z) = z^{-k}h(z)$, *when* $z_j = \infty$ *and* $f(\infty) = \infty$,
with $h(z)$ *analytic and* $h(z_j) \neq 0$, $k \geq 2$. *In other words,* $(\widehat{\mathbb{C}}, f)$ *is a branched covering Riemann surface of* $\widehat{\mathbb{C}}$ *and the branch points are* z_j.

Proof: Since $\lim\limits_{z \to a_j} f(z) = 0$, we can find a positive number r small enough such that the pre-image Γ of the circle γ_r centered at the origin and of radius r will have disjoint components Γ_j, each containing just one zero a_j. If ∞ is a zero of f, then the respective component Γ_0 must be traversed clockwise, in order for ∞ to remain on its left. We understand by the domain bounded by Γ_0 (if Γ_0 exists) that component of $\widehat{\mathbb{C}}$ defined by Γ_0 which contains ∞. For the opposite orientation of Γ_0 we have a curve containing all the other components Γ_j.

Moreover, we can choose the above r such that $f'(z) = 0$ has no solution in the closed domain bounded by Γ_j except maybe for a_j. Then, for an arbitrary $\theta \in R$, the equation $f(z) = re^{i\theta}$ has exactly α_j distinct solutions on Γ_j. Now, consider the pre-image by f of the ray inside γ_r determined by $re^{i\theta}$. In the domain bounded by Γ_j it consists of a union of α_j Jordan arcs having in common only the point a_j and connecting a_j to the solutions of $f(z) = re^{i\theta}$ on Γ_j, $j = 0, 1, 2, ..., p$ (see [A, p. 131– 133]).

Let c_k, $k = 1, 2, ..., m$, be the solutions of the equation $f'(z) = 0$ external to all Γ_j, and let $w_k = f(c_k) = r_k e^{i\theta_k}$. Suppose that $r_1 \leq r_2 \leq ... \leq r_m$. When $r_k = r_{k+1}$, then we take $\theta_k < \theta_{k+1}$, for every k. We perform simultaneous continuation starting from all a_j over a curve L from the w-plane in the following way. We take first the pre-image by f of the segment from 0 to $r_1 e^{i\theta_1}$. This is a union of arcs, α_j of which are starting in a_j, $j = 0, 1, 2, ..., p$. At least one of these arcs is connecting one of the a_j with c_1. If $r_1 = r_2$, then we take the pre-image of the shortest arc between w_1 and w_2 of the circle centered at the origin and having the radius r_1 (if $w_1 = -w_2$, we go counter-clockwise on that circle), etc. If $r_k < r_{k+1}$, we take the pre-image by f of the union of the arc of circle centered at the origin and having the radius r_k, between w_k and $r_k e^{i\theta_{k+1}}$, and the segment between this last point and w_{k+1}. After the point w_m has been reached, if f has at least one multiple pole, we take the pre-image of the ray from w_m to ∞. If f has no multiple pole, then the end of L is w_m and therefore L is a finite path. In this way we build in a few steps the path L and the simultaneous continuation over L starting from all a_j. The continuation arcs can have in common only points a_k, b_k or c_k, and all b_k and c_k are reached by several pre-image arcs. Indeed, if two such arcs meet in a point c, then they are both mapped by f on the same sub-arc of L starting in $f(c)$. One of the following four situations may happen:

a) $f(c) = 0$ and $f'(c) = 0$, hence c coincides with a multiple zero a_k. Then f has the expression (i) with $w_0 = 0$ in a neighborhood of $c = z_j$.

b) $f(c) \neq 0$ and $f'(c) = 0$, hence c coincides with a c_k. Then f has the expression (i) with $w_0 = f(c)$ in a neighborhood of $c = z_j$.

c) $f(c) = \infty$ and c is a multiple pole b_k of f. Then f has the expression (ii) in a neighborhood of $c = b_k = z_j$.

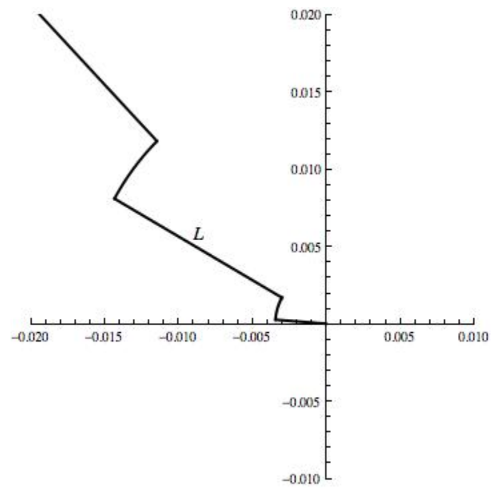
d) $c = \infty$. Then f has the expression (iii) in a neighborhood of ∞.

On the other hand, every c_k and b_k must be reached by some continuation arcs, since $f(c_k) \in L$ and $f(b_k) \in L$. More exactly, there are as many continuation arcs starting in c_k as the multiplicity of c_k as zero of the equation $f'(z) = 0$ and there are as many continuation arcs starting in b_k as the multiplicity of b_k as a pole of f. The arcs starting in simple zeros of f border exactly n

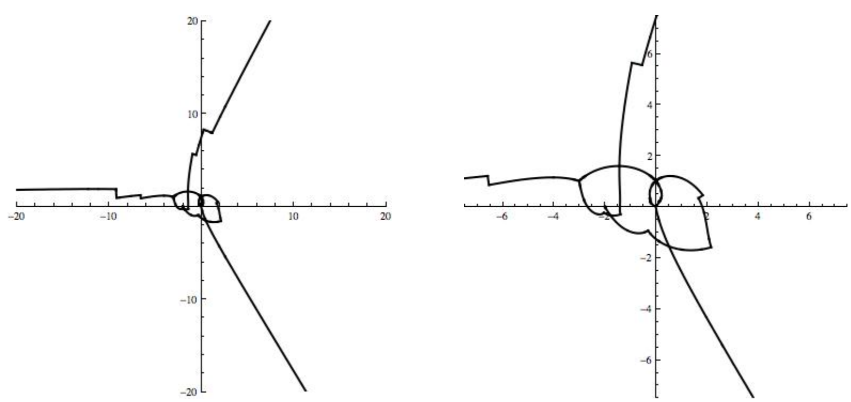
bounded and/or unbounded domains Ω_k (fundamental domains) which are mapped conformally by f on the w-plane from which the curve L has been removed. This is a corollary of the boundary correspondence theorem (see [N, p. 154]). If we denote by $\overline{\Omega}_k$ the closure of Ω_k, then it is obvious that $\widehat{\mathbb{C}} = \cup_{k=1}^{n} \overline{\Omega}_k$. With the notation $A_k = \overline{\Omega}_k \setminus \cup_{j=1}^{k-1} \overline{\Omega}_j$ we have the partition in the statement of the theorem.

Example 5: $f(z) = \dfrac{z^3(z+2)}{(z-i)^4(z+3-i)^3}$

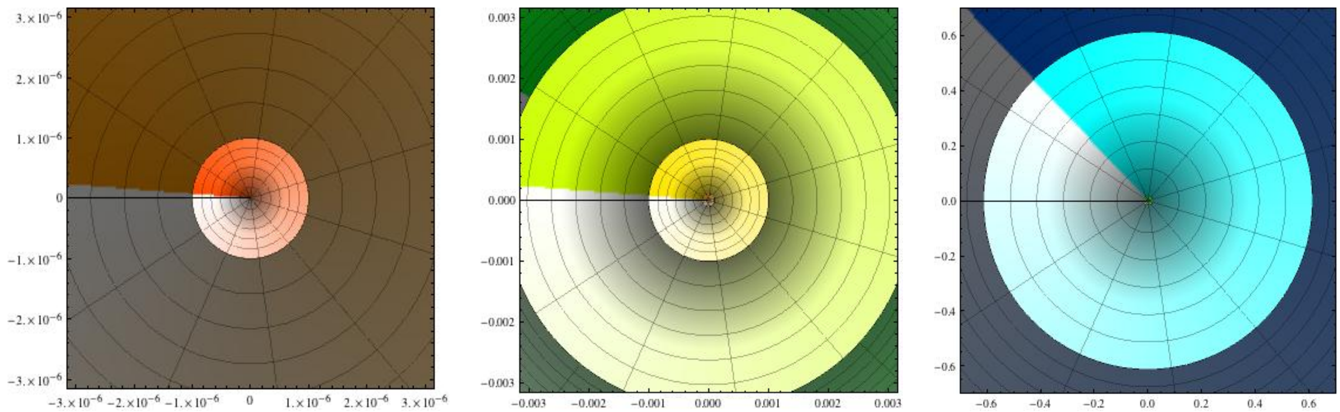
We consider the polygonal line L passing through each image of the zeroes of f'



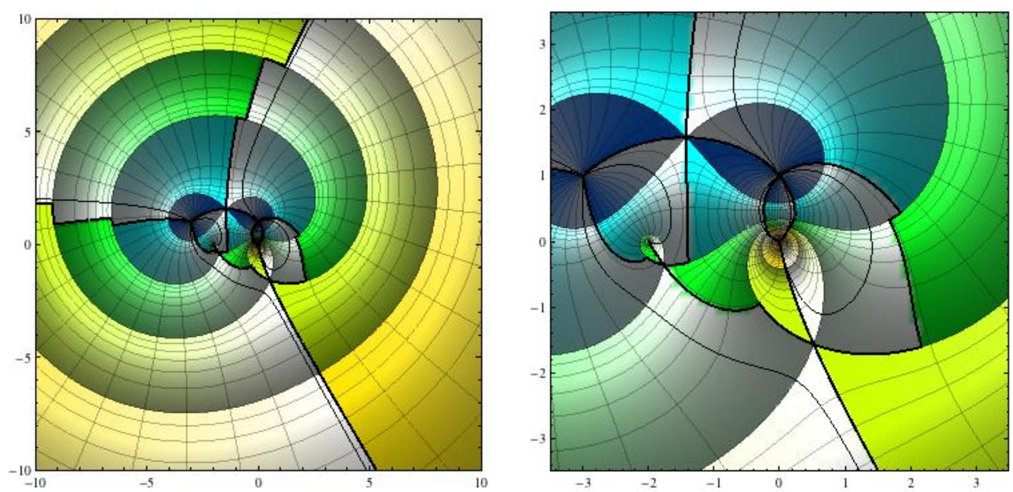
whose pre-image under f is:



Below we consider a collection of colored annuli. In this case the saturation of the annuli increases starting at the polygonal line L.



Their image under f is shown below.

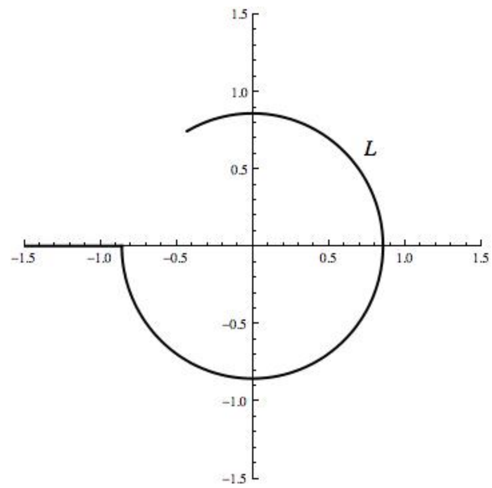


Finally, we examine the case in which f is a polynomial of degree n. Then the unique pole of f is ∞ and it has multiplicity n. Hence, the ray from w_m to ∞ has as pre-image n infinite arcs and all the domains Ω_k are unbounded. For a polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + ... + a_n$, $a_0 \neq 0$, we can describe these infinite arcs. Suppose that $\arg a_0 = \alpha$ and $\arg c_m = \beta$ and let $z_k(t)$, $t > 0$, be the parametric equation of one of these arcs. Then $P(z_k(t)) = a_0[z_k(t)]^n[1 + a_1/z_k(t) + \ldots]$ and $\arg P(z_k(t)) = \beta$. In other words, $\alpha + n \arg z_k(t) + o(t) = \beta + 2j\pi$, $\lim_{t\to\infty} o(t) = 0$. Hence $\lim_{t\to\infty} \arg z_k(t) = \dfrac{\beta - \alpha}{n} + \dfrac{2j\pi}{n}$. Thus, the arcs $z_k(t)$ tend asymptotically to the rays of slope $\dfrac{\beta - \alpha}{n} + \dfrac{2j\pi}{n}$, $j = 0, 1, ..., n-1$. This leads to the following theorem.

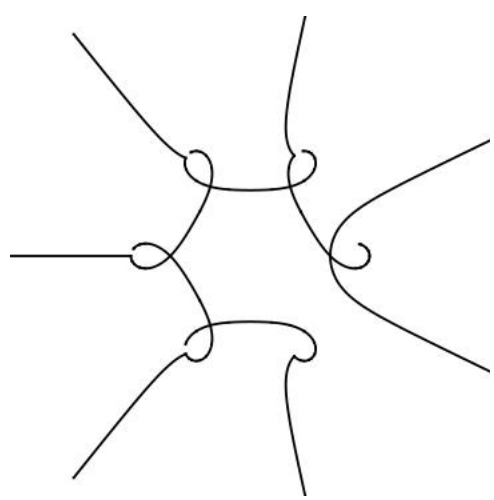
Theorem 3 *Every polynomial P of degree n defines a partition of $\widehat{\mathbb{C}}$ into n unbounded regions such that the interior of every region is mapped conformally by P on $\widehat{\mathbb{C}} \setminus L$, where L is a cut. The mapping can be extended analytically to L, except for a finite number of points, such that $(\widehat{\mathbb{C}}, P)$ is a branched Riemann covering of $\widehat{\mathbb{C}}$ having those points as branch points. The fundamental domains of $(\widehat{\mathbb{C}}, P)$ are bounded by arcs which tend asymptotically to n rays, every two consecutive rays forming an angle of $2\pi/n$.*

Example 6: $P(z) = \dfrac{z^7}{7} - z$, $P'(z) = z^6 - 1$. The branch points are the sixth roots of unity.

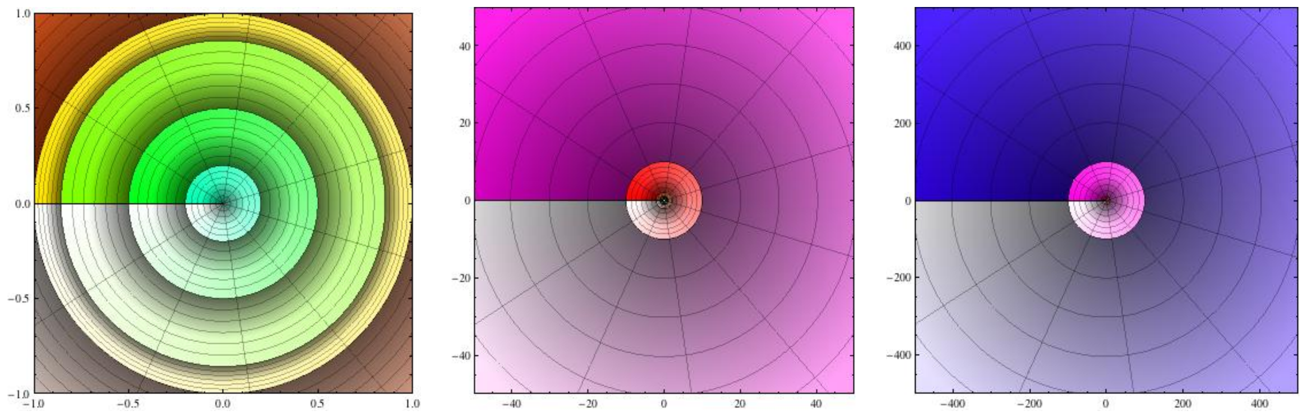
We consider the polygonal line L passing through each image of the zeroes of P'



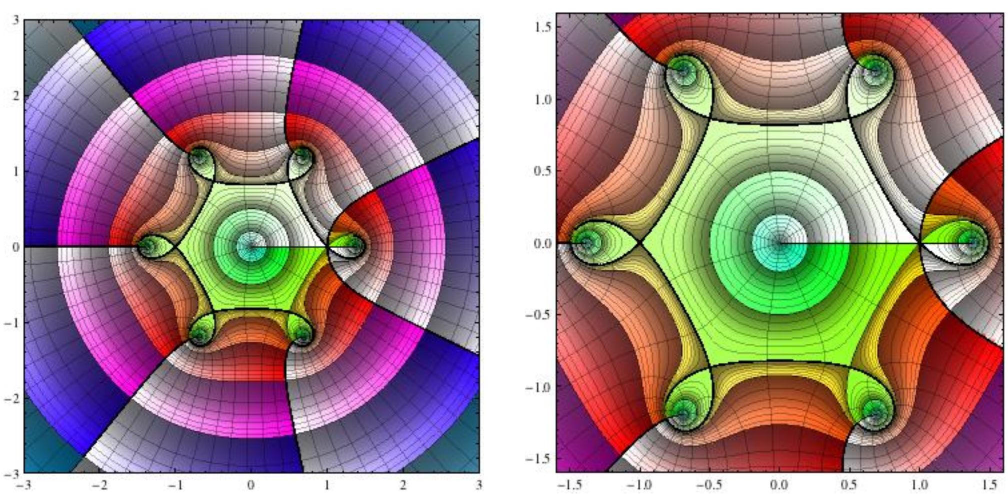
whose pre-image under P is:



The pre-images of the annuli



under P are shown below.



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