Conceptual Anachronism–A Case Study from Interpretations of the *Conics* of Apollonius

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Outline

Who was Apollonius?

- Lived ca. 262–190 BCE, born in Perga (south coast of present-day Turkey)
- Active roughly 75 - 100 years after time of Euclid (ca. 300 BCE); slightly younger than Archimedes (ca. 287–212 BCE)
- Studied with successors of Euclid at *Museum* in Alexandria
- Have lists of his works from later commentaries, but most have not survived
- Know he did astronomy as well, work incorporated into Claudius Ptolemy's (90 - 168 CE) geocentric model of solar system with epicycles, etc.

From Taliaferro's translation – start of Book I

A few historical details can be gleaned from prefatory letters at the start of several books of the *Conics*, e.g. from letter to Eudemus at head of Book I:

... I worked out the plan for these conics at the request of Naucrates, the geometer, at the time he was with us in Alexandria lecturing, and ... on arranging them in eight books, we immediately communicated them in great haste because of his near departure, not revising them but putting down whatever came to us ... it happened that some others among those frequenting us got acquainted with the first and second books before the revision ...

Previous work on conics

- Some historians attribute the first work on conics to Menaechmus (ca. 380 - 320 BCE; in Plato's circle)
- But conics only feature in his solutions of the "Delian problem" – duplication of the cube – using parabolas and/or hyperbolas
- (More) systematic work on conics by Aristaeus (before the time of Euclid) and Euclid (ca. 300 BCE) himself
- Those older works are known now only through comments made by Pappus of Alexandria (ca. 300 - 350 CE) in his *Collection* – a sort of summary and encyclopedia of much of the Greek mathematics of the classical and Hellenistic periods – and the commentaries on Euclid by the Neoplatonist philosopher Proclus (412 - 485 CE)

Theory of conic sections before Apollonius

- On the basis of some details preserved, it's thought that the earlier work dealt only with *right cones* generated by rotating a right triangle about one of its legs, and slicing by planes perpendicular to the other leg.
- Note that then the *vertex angle* of the cone determines the type of the section obtained:
- In later terminology: acute angles – ellipses; right angles – parabolas; obtuse angles – hyperbolas
- Apollonius discusses some aspects of this earlier work in rather disparaging ways at several points; judging from the tone, he seems (to me at least) to have had a "prickly" streak(!)

Archimedes and conics

- Archimedes is probably the best-known of the Greek mathematicians following Euclid (certainly more commonly read than Apollonius – more accessible in several ways!)
- His work also involves conic sections to a large degree, especially the *Quadrature of the Parabola*
- He does not use the Apollonian terminology
- calls a parabola a “section of a right-angled cone”

Plan of Apollonius' *Conics*

- Books I, II, III, IV – "Elements of conics" – definitions and basic properties; properties of asymptotes of hyperbolas; tangents; intersections of conics
- The above survive in original Greek versions. The following are only known through later Arabic translations:
- Book V, VI, VII – "Researches on conics" – normals to conics, maximum and minimum distances from a point, the evolute of a conic (or things that can be interpreted in those terms!), equality and similarity of conics, "limiting properties"
- Book VIII – ? (lost – several attempts at "reconstruction" including one famous one by E. Halley, 1710 CE)

Apollonius' framework is more general

- Definition 1 from Book I: *If, from an arbitrary point, a line is drawn to the perimeter of a circle which does not lie in the same plane with the point and extended indefinitely in both directions, and with the point remaining fixed, the line is moved around the circle back to its starting position, then the surface described by the line, which consists of two surfaces both containing the fixed point, ... I will call a conic surface; I will call the fixed point the vertex of the conic surface and the line through the vertex and the center of the circle I will call the axis.* (translation by JBL, close to literal)
- A *cone* is the figure bounded by a nappe of the conic surface and a plane parallel to the circle's plane.

Apollonius' Definition 4 – anachronism?

- Apollonius defines a *diameter* of a plane curve in his Definition 4: *a straight line that bisects all the straight lines between pairs of points of the curve, drawn parallel to some given straight line, is called a diameter of the curve*; endpoints of a diameter are *vertices* of the curve.
- Some standard English translations of Apollonius (e.g. Heath, Taliaferro, ...) say those parallels have been drawn *"ordinatewise"* to the diameter.
- The actual Greek phrase is used repeatedly in the *Conics*, so deserves special consideration: literally means something more like *"lined-up"* or *"in order"*, or drawn *"in an orderly fashion"*; we'll return to this next time.

The first propositions in Book I – (not literal translations)

- **Proposition 1.** If a straight line through the vertex of a conic surface meets the surface at some other point, then the line lies in the conic surface.
- **Corollary.** If a straight line through the vertex of a conic surface contains a point inside (resp. outside) the surface, then the line lies inside (resp. outside) the surface.
- **Proposition 2.** If two points lie on one of the nappes of the conic surface, the line segment joining them will not pass through the vertex and will lie inside the conic surface; the rest of the line containing the two points will lie outside the surface.

First propositions, continued

- **Proposition 3.** If a cone is cut by a plane through the vertex, the section is a triangle. (Recall, a *cone* is the solid figure bounded by the conic surface and a plane parallel to the plane of the generating circle. Also, if the plane contains the axis of the cone, the resulting triangle is called an *axial triangle*.)
- **Proposition 4.** If either one of the nappes of the conic surface is cut by a plane parallel to the plane containing the generating circle, the intersection is a circle with its center on the axis of the cone, and the figure bounded by nappe and that circle is also a cone.
- Proposition 5: oblique (non-right) conic surfaces have “subcontrary” circular sections in addition to the sections in Proposition 4.

Proposition 6 (Taliaferro's translation)

If a cone is cut by a plane through the axis, and if on the surface of the cone, some point is taken which is not on a side of the axial triangle, and if from this point is drawn a straight line parallel to some straight line which is a perpendicular from the circumference of the circle to the base of the triangle, then that drawn straight line meets the axial triangle, and on being produced to the other side of the surface the drawn straight line will be bisected by the triangle.

Figure for Proposition 6

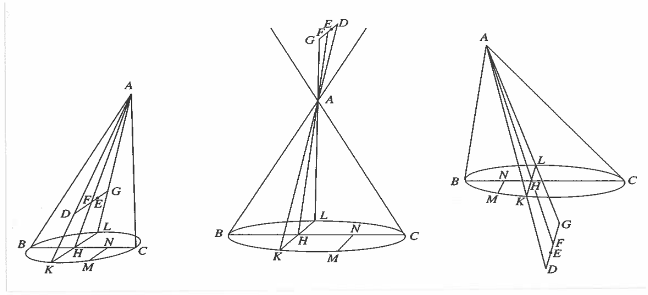


Figure: From Green Lion Press edition of Taliaferro's translation

Proposition 7 – diameters

Apollonius uses Proposition 6 to prove the following (translation by JBL, simplified and not literal)

Proposition (7)

If a cone is cut by plane through its axis, and by a second plane that intersects the plane of the base along a line perpendicular to the base of the axial triangle, then the intersection of the plane of the axial triangle and the plane of the section is a diameter for the section.

He notes that the parallels bisected by the diameter need not be perpendicular to it if the cone is not a right cone.

Figure for Proposition 7

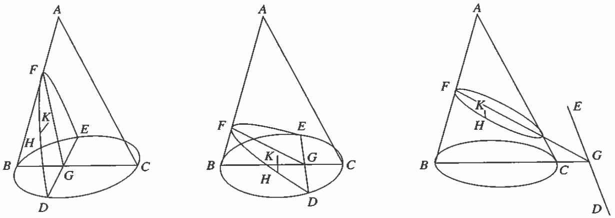


Figure: From Green Lion Press edition of Taliaferro's translation

Setting up

- **Proposition 8.** If, in the situation of Proposition 7, the resulting diameter of the section is parallel to one of the sides of the axial triangle, then the section "will increase indefinitely" (i.e. it is not a closed curve)
- **Proposition 9.** If a cone is cut by a plane which meets all three sides of the axial triangle (produced) and is neither parallel to the plane of the base, nor subcontrary, then the section will not be a circle.
- **Proposition 10.** If two points are taken on the section of a cone, the straight line joining the two points will fall inside the section, and produced will fall outside.

The next propositions

- Apollonius classifies the conic sections according to the way the cone is sectioned
- Derives from that their *sumptomata* ("fundamental properties")
- For future reference – these *sumptomata* are expressed in each case as a relation between a given square and a given rectangle
- Constructed from an arbitrary point on the curve, together with an auxiliary fixed line segment (the *orthia pleura*, "*latus rectum*," upright side)

Proposition 11 – incorporates definition of the parabola (Taliaferro's translation)

If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if, further, the diameter of the section is parallel to one side of the axial triangle, and if any straight line is drawn from the section of the cone to its diameter such that this straight line is parallel to the common section of the cutting plane and of the cone's base, then this straight line to the diameter will equal in square the rectangle contained by (a) the straight line from the section's vertex to where the straight line to the diameter cuts it off and (b) another straight line which has the same ratio to the straight line between the angle of the cone and the vertex of the section as the square on the base of the axial triangle has to the rectangle contained by the remaining two sides of the triangle. And let such a section be called a parabola.

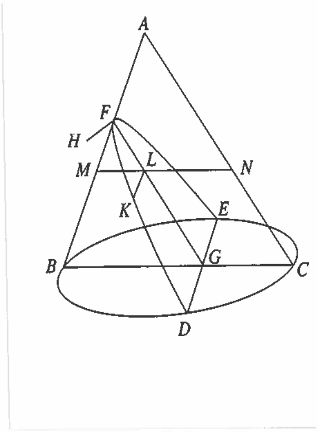
Some tendentious comments

- As you might guess from this sample, Apollonius' prose is verbose, complicated in syntax, *and mathematically dense* – a "hard slog" (in the original Greek, or in translation)!
- J. Kepler in response to criticism of his own works: *If anyone thinks that the obscurity of this presentation arises from the perplexity of my mind, ... I urge any such person to read the Conics of Apollonius. He will see that there are some matters which no mind, however gifted, can present in such a way as to be understood in a cursory reading. There is need of meditation, and a close thinking through of what is said.*
- Are those in the "Twitter generation" capable of this?

Structure of a Euclidean or Apollonian proposition

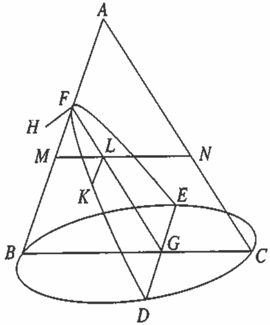
- Probably time for a digression on the structure of the propositions in Apollonius's text (follows Euclidean model closely)
- Typically, there are 6 "parts" – *protasis*, [diagram and] *ekthesis*, *diorismos*, *kataskeuē*, *apodeixis*, *sumperasma*
- The supremely complex and convoluted first sentence on the previous slide is the *protasis* of this proposition – the "statement"
- The *ekthesis* then "lays out" the statement by means of a figure and the usual sort of labeling of important points with letters.

Ekthesis of Proposition 11, beginning (condensed translation by JBL)



Let A be the vertex, ΔABC the axial triangle, and let the other plane cut the plane of the base in DE perpendicular to BC. The section is the curve DFE, with diameter FG parallel to AC.

Conclusion of *ekthesis* and *diorismos* of Proposition 11



Let H be "contrived so that"

$$(*) \quad sq.BC : rect.BA, AC :: FH : FA$$

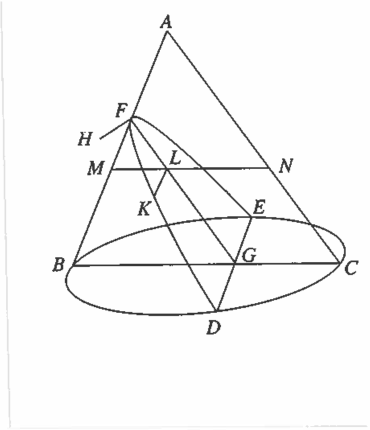
Finally let K be taken at random on the section and let KL be parallel to DE.

I say that $sq.KL = rect.HF, FL$.

A word on terminology and notation

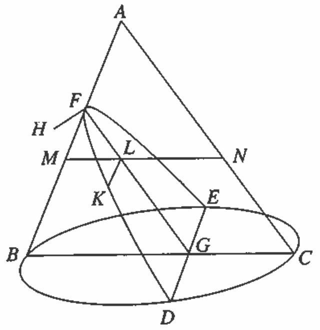
- The notation here is Taliaferro's modern attempt to capture what Apollonius actually says in a (more) readable way
- Apollonius' Greek is highly conventionalized and abbreviated, but entirely expressed in words
- Here $sq.XY$ means (the area of) the square with side XY (Apollonius in fact just says literally "the from XY")
- $rect.XY, YZ$ stands for (the area of) the rectangle with sides XY and YZ (Apollonius in fact just says literally "the by XYZ")
- The $:$ and $::$ are standard notation for comparing ratios (Books V, X of Euclid contain an exposition of Eudoxus' theory of these)

Kataskeuē and start of *apodeixis* of Proposition 11



For let the straight line *MN* be drawn through *L* parallel to *BC*. Since *DE* is also parallel to *KL*, the plane through *KL* and *MN* is parallel to the plane of the base and by Proposition 4, the section in that plane (containing *M*, *K*, *N*) is a circle and *MN* is a diameter.

Apodeixis of Proposition 11, continued



Then *Elements* Books III and VI imply $rect.ML, LN = sq.KL$. Combining this with (*) ("*componendo*")

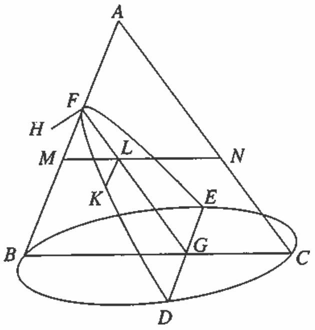
$$HF : FA :: BC : CA \text{ cp. } BC : BA$$

But by similar triangles

$$BC : CA :: MN : NA :: ML : LF,$$

$$BC : BA :: MN : MA :: NL : FA$$

Apodeixis of Proposition 11, concluded



So substituting,

$$HF : FA :: ML : LF \text{ cp. } NL : FA$$

But with FL as the common height

$$HF : FA :: rect.HF, FL : rect.LF, FA$$

So $rect.ML, LN = rect.HF, FL$ and substituting

$$sq.KL = rect.HF, FL.$$

(Euclid would add *oper edei deixai*, but Apollonius doesn't.)

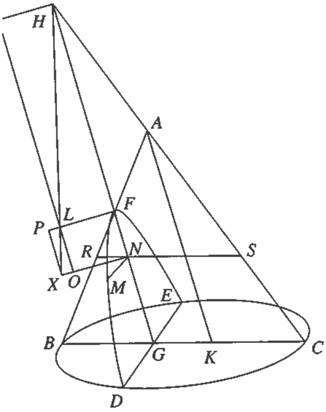
Historical comments

- This fact about parabolas was *certainly not new* in the work of Apollonius (almost nothing in Books I - IV probably is)
- Archimedes' *Quadrature of the Parabola*, for instance, states something very close, precisely – if KL and $K'L'$ are *two* such segments, then

$$sq.KL : sq.K'L' :: FL : FL',$$

  and it's given *without proof*
- Usual interpretation (almost certainly correct) is that Archimedes took this from some standard reference of his time on conics, probably the lost Euclid *Conics*
- What seems to be new in Apollonius is the realization that all three families of conics can be obtained as sections of *any right or oblique cone*.

Conclusion of *Ekthesis* and *diorismos* of Proposition 12 – the hyperbola

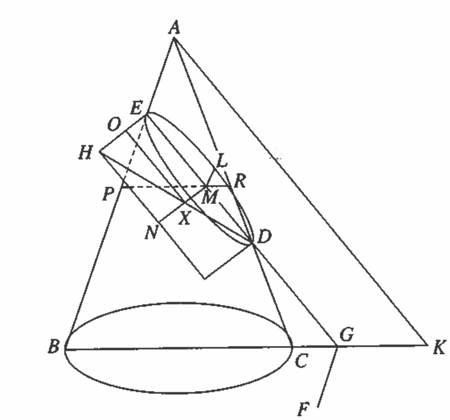


Let $AK \| FG$, $FL \perp FG$ and let it "be contrived that"

$sq.KA : rect.BK, KC :: HF : FL.$

I say that MN is equal in square to the rectangle FX which *exceeds* FO by LX, similar to the rectangle contained by HF, FL.

Conclusion of *Ekthesis* and *diorismos* of Proposition 13 – the ellipse



Let $AK \| ED$, let $EH \perp ED$ and let it "be contrived that"

$sq.AK : rect.BK, KC :: DE : EH.$

I say that LM is equal in square to the rectangle EX–the rectangle containing HE, EL, *less, or deficient HX* similar to the rectangle contained by DE, EH.

The names "parabola," "hyperbola" and "ellipse"

- Greek mathematical terminology often "borrowed" common words and gave them special meanings.
- Apollonius did this here (following earlier work in a different context – "application of areas")
- *parabolē* – noun: a "throwing alongside," comparison, juxtaposition
- *hyperbolē* – noun: a "throwing beyond," excess, superiority
- *elleipō* – verb: to fall short, be in want of, lack

Reconsidering the term "ordinatewise"

- Greek mathematics (like ours!) had a technical vocabulary; common words but with specialized meanings
- So we should be on the lookout not to rely too much on common meanings
- But as far as we know Apollonius invented this formulation
- Interestingly enough, the entry in the standard LSJ Greek lexicon for *tetagmenōs* (the word translated as "ordinatewise") gives the common meaning and then "ordinatewise" with a reference to Definition 4 in Book 1 of Apollonius(!)
- A guess: some mathematician *(maybe Heath?)* provided this citation to the compilers of the lexicon(!)

tetagmen$\bar{o}$s to "ordinatewise?"

- First Latin translation of Apollonius to circulate widely in Western Europe by Federigo Commandino (1509-1575 CE); then several others too, including the one by Halley
- Commandino's Latin rendering: *ordinatim applicatae* – pretty literal version of the *everyday Greek meaning of the word – "applied in an orderly fashion."* Was the English term derived later from this? (And by whom?)
- N.B. in the meantime Descartes' *La Géométrie* published in 1637 CE – slightly old-fashioned analytic geometry terminology: "abcissas and ordinates" are x and y coordinates(!)
- So was Apollonius was thinking in coordinate terms?? Gets very subtle(!)

Descartes' view

From *La Géométrie*:

... je vous prie de remarquer en passant que le scrupule que faisoient les anciens d'user des termes de l'arithmétique en la géométrie, qui ne pouvoit procéder que de ce qu'ils ne voyoient pas assez clairement leur rapport, causoit beaucoup d'obscurité et d'embarras en la façon dont ils s'expliquoient

Apollonius' "equation of a parabola?"

- Recall the *diorismos* of Apollonius' Proposition 11: "I say that $sq.KL = rect.HF, FL$."
- Segments like *KL* are said to be drawn "ordinatewise" (mis?)reading Commandino
- If we write $y = \overline{KL}$ and $x = \overline{FL}$ (the corresponding "abcissas"), then noting that *HF* is a fixed segment of length *c*, say, our equation just gives the "sideways" parabola $y^2 = cx$ (and *c* corresponds to the modern *latus rectum*)
- Can get analogous statements for the hyperbola and ellipse as well, of course!

Apollonius' proof of Proposition 11, reconsidered

- At some point after Descartes' analytic geometry had become a standard tool, it became almost "too easy" to interpret parts of Greek mathematics in algebraic terms
- Proofs like Apollonius' Proposition I.11 can easily be translated as manipulations of fractions and equations and we get statements that look very modern
- "*The work of Apollonius in many respects approaches so closely to the modern form of treatment that it not infrequently has been regarded as constituting analytic geometry*." (C. Boyer, History of Analytic Geometry)

Zeuthen and Heath

- For Apollonius: especially H. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, first ed. 1886
- Apollonius refers to propositions in Euclid's Book II many times and Zeuthen was very influential in the way people read *Conics* and also Euclid's *Elements*, Book II
- Described as "geometric algebra" (and eventually conjecturally linked to earlier Babylonian texts by O. Neugebauer)
- Heath's version of Apollonius ("edited in modern notation"), first ed. 1896 – says that his aim was to remedy the relative neglect of Apollonius by providing a version "*so entirely remodelled by the aid of accepted modern notation as to be thoroughly readable by any competent mathematician*"

Geometric algebra?

Proposition 5 in Book II of the *Elements*: *If a straight line is cut into equal and unequal [pieces], then the rectangle contained by the unequal pieces of the whole [line], plus the square on the [difference] between the pieces is equal to the square on half [of the line]*.

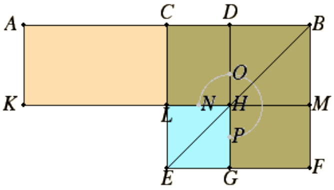


Figure: Stolen from:
`http://aleph0.clarku.edu/~djoyce/elements`

More ambiguous than you might think

- For us as modern mathematicians, it's natural (probably *almost too difficult not to*) translate these statements into algebra(!)
- For instance in Euclid II.5, if we make $x = AC = BC$, $y = CD$ we have an equivalent statement

$$(x + y)(x - y) + y^2 = x^2$$

- But also, if we let $x = AD$ and $y = BD$ so $x > y$, then Euclid's statement is also equivalent to

$$xy + \left(\frac{x - y}{2}\right)^2 = \left(\frac{x + y}{2}\right)^2$$

- Which algebraic version was Euclid thinking of? Does it matter?

A contrary view

- S. Unguru (1931-present CE), "On the need to rewrite the history of Greek mathematics," Archive for History of Exact Sciences 15 (1975/76), 67–114.
- Forcefully refutes "geometric algebra" as a correct description of Book II of Euclid and the use of algebraic reformulations of Greek geometrical texts
- In a 2001 book on Apollonius with M. Fried, he extends this to Apollonius' *Conics*
- Unguru's main point: it's geometry pure and simple; Greek mathematics did not have any of the apparatus of symbolic algebra

Heath "for the defense" on his choice of presentation

- It should be Apollonius and nothing but Apollonius, and nothing should be altered in the substance or in the order of his thought
- It should be complete, leaving out nothing of any significance or importance
- It should exhibit under different headings the successive divisions of the subject, so that the definite scheme followed by the author may be seen as a whole
- Apollonius' method "*does not essentially differ from that of modern analytic geometry except that in Apollonius geometrical operations take the place of algebraical calculations*"

Unguru's point, expanded

- Attempting to "explain" Apollonius or Euclid this way is *perniciously wrong* from the historical point of view
- It uses modern concepts that are a false description of a fundamentally different understanding of mathematics
- *Conceptual anachronism* or "Whig history" – presents the past as leading inevitably to the present
- In particular: in symbolic algebra, variables are effectively placeholders for number values
- But for the Greeks, "number" (*arithmos*) always meant a "counting number" (a positive integer)

Unguru's point, continued

- So, Apollonius (following Euclid) *never* uses a numerical value as a measure of length or area
- Thought by many for a long time to be an after-effect of the discovery of incommensurable ratios ($=$ irrational numbers) earlier, but that too – or at least the idea of a Greek "crisis in foundations" – has now been questioned(!)
- And in the "arithmetical" books VII–IX of the *Elements*, Euclid essentially used geometry to understand properties of (his) numbers, not the other way around(!)

If Unguru wanted to start a war, he succeeded!

A fairly typical example of the tone of Unguru's 1975 article–really quite extraordinary:

" ... history of mathematics has been typically written by mathematicians ... who have either reached retirement age and ceased to be productive in their own specialties or become otherwise professionally sterile. ... The reader may judge for himself how wise a decision it is for a professional to start writing the history of his discipline when his only calling lies in professional senility."

A “flame war,” part 1

- When Unguru’s 1975 article appeared, one of the mathematicians/historians who had championed the idea of Greek “geometric algebra” (and whom Unguru had savaged), B. L. van der Waerden, was still alive (his dates: 1903–1996 CE).
- You can imagine how well he liked that passage from Unguru’s article!
- He published a rejoinder–a defence of his point of view in the same issue of the Archive for History of Exact Sciences – “A defence of a ‘shocking’ point of view,” 199–210. (The “shocking” was Unguru’s characterization(!))

"Flame war," parts 2 and 3

H. Freudenthal: "What is algebra and what has it been in history?" Archive for History of Exact Sciences 16 (1976/77), 189–200. Argues that there is indeed algebra in Greek mathematics, using examples from Archimedes. But of course, a historian would say "I thought we were talking about Euclid and/or Apollonius" (quite different)

A. Weil: "Who betrayed Euclid? Extract from a letter to the editor," Archive for History of Exact Science 19 (1978/79), 91–93. Essentially asks: "who was responsible for allowing such a trashy, polemical article to be published? What is happening to the quality of this journal?" And gets in a nice ad hominem attack: "... it is well to know mathematics before concerning oneself with its history ... "

What can we learn from this?

- This may all strike you as a nasty but silly disagreement over a rather minor issue
- But it points out a fundamental difference between doing mathematics and doing history of mathematics (as history)
- As mathematicians, recognizing logical connections between old and new work and making reinterpretations is a part of what we do.
- When apparently different things are logically the same, just expressed in different ways, we can and do treat them as the same(!) And we are always looking for those equivalences–finding them can represent an advance in our understanding!

And maybe Unguru had a point?

- (As Unguru insinuated in his own nasty way) Zeuthen, van der Waerden, Freudenthal, Weil, etc. were certainly all primarily mathematicians who had eminent research records
- Then turned to writing history later in their professional careers
- Not surprising that they had the "habits of mind" and point of view of working mathematicians, not historians!
- In particular, to put words in their mouths: "if it's logically equivalent to algebra, but expressed in geometric terms, then it's a geometric form of algebra"

The "take-home" message

- For intellectual historians, not so much logical equivalences that matter–it's particular features, differences!
- Each culture, era, scientific school, etc. is a unique and separate thing
- Unguru: The mathematical historian's first and most important job is to understand a body of mathematical work on its own terms, *not on our terms*
- A fundamentally different way of thinking
- Recent article by K. Saito: "Mathematical Reconstructions Out, Textual Studies In" summarizes current state of mathematical historiography

With some additional moderation

- In a recent article, "Apollonius, Davidoff, Rorty, and Zeuthen: From A to Z, what else is there?" (Sudhoffs Archiv, 91 (2007), 1 - 19), Unguru and Fried make it clearer that their "issues" concern Zeuthen's work *qua history*, not *qua* mathematics
- and they contrast Zeuthen's well-intentioned and mathematically astute (mis)reading with a parodied "post-modern," deconstructionist view that would deny any intrinsic meaning in a text
- make their point via a (hilarious, fictional) "sexual politics reading" of Apollonius

So what can we say about Apollonius?

- It's a geometrical masterwork containing some amazing stuff, but *it's not* coordinate geometry (didn't exist yet – and while he might relate lengths measured to two lines in Propositions 11,12,13, he never uses coordinates on the whole plane)
- But we do know his work (and summaries in Pappus) was read very carefully by Descartes and others – much of *La Géométrie*, for instance, is devoted to discussion of a problem going back to Apollonius and Euclid by way of Pappus – Descartes uses his methods to provide a superior solution
- The methods Apollonius developed certainly did provide a big stimulus to the development of analytic geometry