

**College of the Holy Cross, Fall Semester, 2005**  
**Math 131, Midterm 3**  
**Solutions**  
**Wednesday, November 30, 6 PM**

1. Compute the derivative of each of the following functions. Simplify your answers as much as possible.

(a) [4 points]  $f(t) = \cos(3t^2 + 5)$

**Solution:**  $f'(t) = -\sin(3t^2 + 5)6t$

(b) [4 points]  $g(x) = \sinh^2(x) - \cosh^2(x)$

**Solution:** Since  $\sinh^2(x) - \cosh^2(x) = -1$ ,  $g'(x) = 0$ . Or take the derivative

$$g'(x) = 2 \sinh(x) \cosh(x) - 2 \cosh(x) \sinh(x) = 0$$

(c) [5 points]  $h(s) = \ln(4s^2 + 1) + \arctan(2s)$

**Solution:**  $\frac{1}{4s^2 + 1}8s + \frac{1}{1 + 4s^2}2 = \frac{8s + 2}{4s^2 + 1}$

2. (a) [9 points] Use implicit differentiation to find the equation of the tangent line to the curve  $y^3 + 2x^2y + x^2 = 13$  at the point  $(2, 1)$ .

**Solution:**  $3y^2y' + 4xy + 2x^2y' + 2x = 0$ ;  $(3y^2 + 2x^2)y' = -2x - 4xy$ ;  $y' = \frac{-2x - 4xy}{3y^2 + 2x^2}$

At  $(2, 1)$ ,  $y' = \frac{-4 - 8}{3 + 2 \cdot 4} = -\frac{12}{11}$ .

The equation of the tangent line is  $y - 1 = -\frac{12}{11}(x - 2)$ . This is the same as  $y = -\frac{12}{11}x + \frac{35}{11}$ .

Or you can use the slope to write  $y = -\frac{12}{11}x + b$  for the equation of the tangent line and use the fact that the point  $(2, 1)$  is on the line to find  $b$ . We have  $1 = -\frac{12}{11}2 + b$ . Thus  $b = \frac{35}{11}$  and  $y = -\frac{12}{11}x + \frac{35}{11}$ .

- (b) [6 points] Find the derivative of the function  $y = x^{e^x}$ .

**Solution:** First take  $\ln$  of both sides:  $\ln y = e^x \ln x$ .

Then take derivatives of both sides:  $\frac{1}{y}y' = e^x \ln x + e^x \frac{1}{x}$ .

Then solve for  $y'$ . We get  $y' = y(e^x \ln x + e^x \frac{1}{x}) = x^{e^x} (e^x \ln x + e^x \frac{1}{x})$ .

3. Evaluate the following limits.

(a) [5 points]  $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{1 - \cos x}$

**Solution:** Since both the numerator and the denominator approach 0 as  $x \rightarrow 0$ , we apply the rule of L'Hospital.

$$\lim_{x \rightarrow 0} \frac{\sin(x^2)}{1 - \cos x} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{\cos(x^2) \cdot 2x}{\sin x} \stackrel{L'H}{=} \lim_{x \rightarrow 0} \frac{-\sin(x^2) \cdot (2x)^2}{\cos x} = 2$$

(b) [5 points]  $\lim_{x \rightarrow 0} \frac{\cos(x^2)}{1 - \sin x}$

**Solution:** Since both the numerator and the denominator approach 1 as  $x \rightarrow 0$ ,

we have  $\lim_{x \rightarrow 0} \frac{\cos(x^2)}{1 - \sin x} = 1$

(c) [5 points]  $\lim_{x \rightarrow \infty} x^2 e^{-x}$

**Solution:**  $\lim_{x \rightarrow \infty} x^2 e^{-x} = \lim_{x \rightarrow \infty} \frac{x^2}{e^x} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{2x}{e^x} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$

4. Consider the function  $f(x) = \frac{x}{x^2 + 1}$

(a) [5 points] Find all of the critical points of  $f$ .

**Solution:**  $f'(x) = \frac{x^2 + 1 - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = \frac{(x - 1)(x + 1)}{(x^2 + 1)^2}$ .

We have  $f'(x) = 0$  when  $x = 1$  or  $x = -1$ . Thus the critical points are  $x = 1$  and  $x = -1$ .

(b) [5 points] Find all local maxima and minima of  $f$ .

**Solution:** Since  $f'(x)$  is negative for  $-\infty < x < -1$  and  $1 < x < \infty$  and positive for  $-1 < x < 1$ , by the first derivative test,  $f$  has a local minimum at  $x = -1$  and a local maximum at  $x = 1$ .

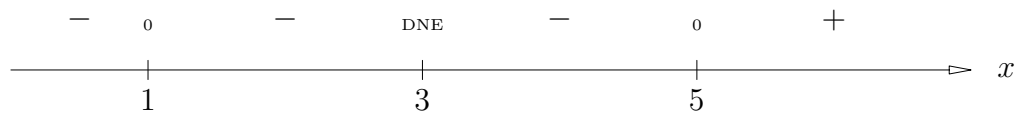
(c) [5 points] Find all points of inflection of  $f$ .

**Solution:**  $f''(x) = \frac{-2x(x^2 + 1)^2 - (1 - x^2) \cdot 2(x^2 + 1) \cdot 2x}{(x^2 + 1)^4} =$   
 $= \frac{-2x(x^2 + 1)(x^2 + 1 + 2 - 2x^2)}{(x^2 + 1)^4} = \frac{-2x(3 - x^2)}{(x^2 + 1)^3}$

$f''(x) = 0$  when  $2x(3 - x^2) = 0$ , i.e. when  $x = -\sqrt{3}, 0, \sqrt{3}$ . By checking values of  $f''$  we see that  $f''$  changes sign at these values and thus  $x = -\sqrt{3}, 0, \sqrt{3}$  are the points of inflection for  $f$ .

5. [12 points] Let  $f$  be a function that has a vertical asymptote at  $x = 3$  and the following sign charts. (The labels DNE indicate that the derivatives do not exist at  $x = 3$ .)

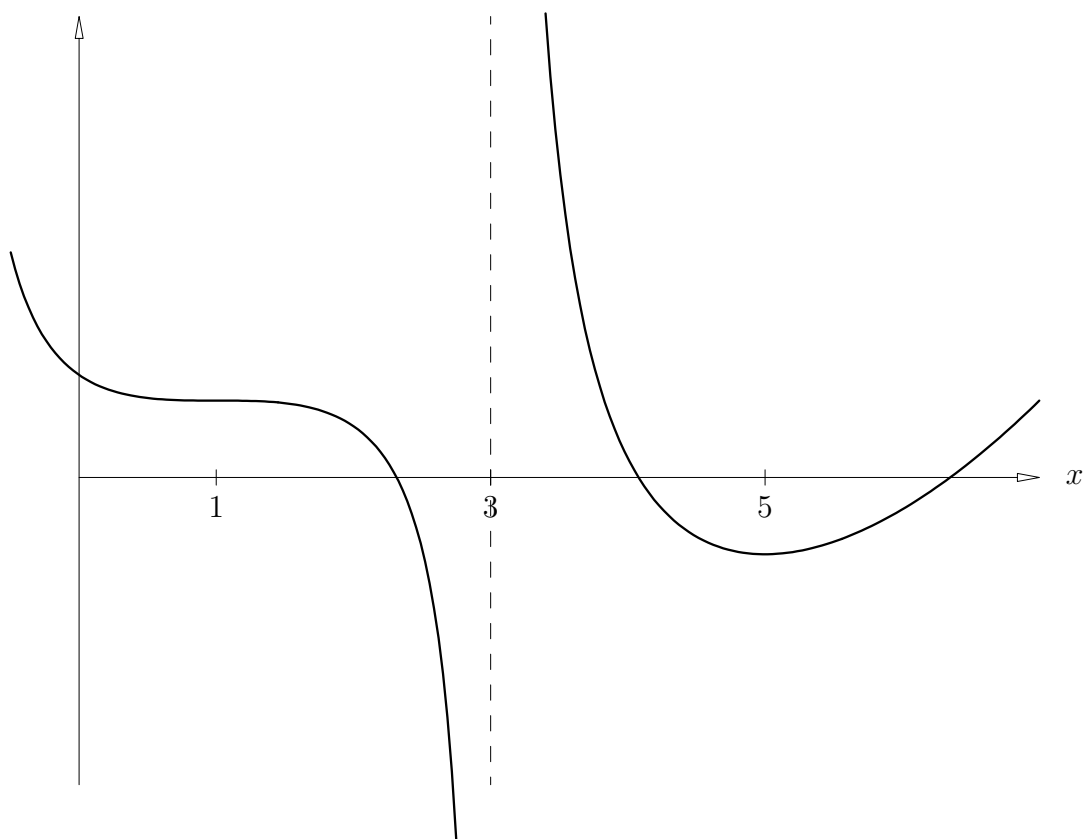
Sign of  $f'$  :



Sign of  $f''$  :

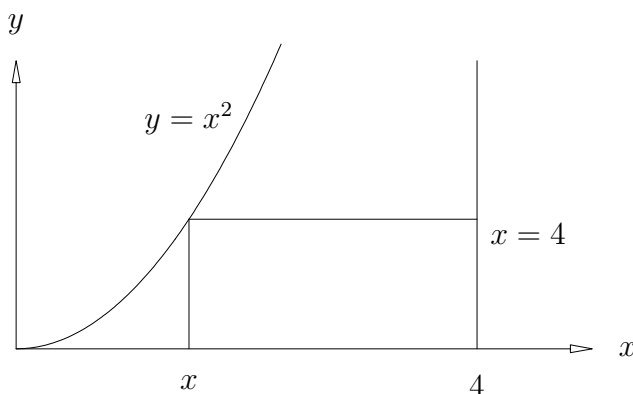


Draw a possible sketch of the graph of  $f$  in the space provided below.



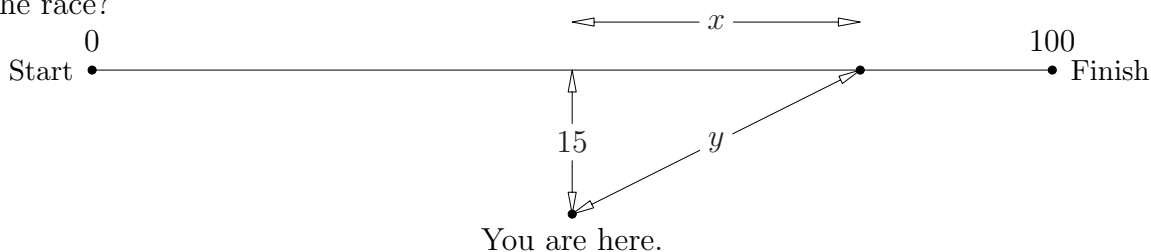
6. [15 points] A rectangle with its base on the  $x$ -axis is inscribed in the region bounded by the curve  $y = x^2$ , the  $x$ -axis and the line  $x = 4$ . Thus, the upper-left vertex is on the curve  $y = x^2$  and the upper-right vertex is on the line  $x = 4$ . Find the dimensions of the rectangle with maximal area.

**Solution:** From the picture, we see that the rectangle has base  $4 - x$  and height  $x^2$ , so  $A(x) = (4 - x)x^2$ ,  $0 \leq x \leq 4$ .



$A'(x) = x^2 + 2x(4 - x) = -3x^2 + 8 = x(8 - 3x)$ . The critical points are  $x = 0$  and  $x = 8/3$ . We have  $A(0) = 0$ ,  $A(4) = 0$  and  $A(8/3) = \frac{256}{27}$ . We see that the maximum area is achieved when  $x = 8/3$ . Thus the dimensions of the rectangle with maximal area are  $4/3$  and  $64/9$ .

7. [15 points] You are at a Holy Cross track meet watching the 100 meter dash, sitting in the bleachers 15 meters back from the midway point of the race. Your friend the speedster is running down the track at a constant speed of 9 meters per second. How fast is the distance between you and your friend changing when she is 80 meters into the race?



**Solution:** Let  $x(t)$  represent the position of your friend at time  $t$  measured from the midpoint. When  $t = 0$  your friend is 50 meters into the race. We have  $\frac{dx}{dt} = 9$ . By Pythagoras, the distance between you and your friend at time  $t$  is given by  $y(t) = \sqrt{15^2 + x(t)^2}$ . Then,  $\frac{dy}{dt} = 1/2(15^2 + x^2)^{-1/2} \cdot 2x \cdot \frac{dx}{dt}$ . When  $x(t) = 30$ , we have  $\frac{dy}{dt} = (15^2 + 30^2)^{-1/2} \cdot 30 \cdot 9 = \frac{18}{\sqrt{5}} \approx 8.05 \frac{m}{sec}$ .