Math 241

Quiz 5 Sample
April 15, 2008

Show all your work on the quiz paper.

1. Let S be the closed set defined by $S = \{(x, y) : 5x^2 - 4xy + 8y^2 \leq 1\}$. The boundary curve B of S is shown in the accompanying diagram. Find the extreme values of $f(x, y) = 3x + y - 1$ on the boundary curve B.

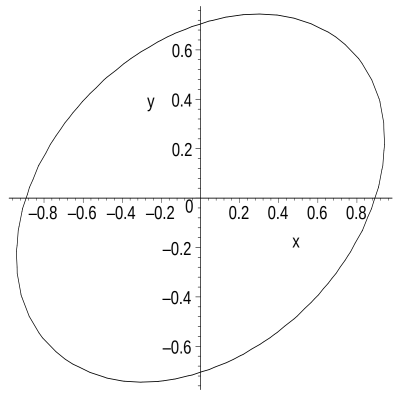

Figure 1:

The constraint function is $g(x, y) = 5x^2 - 4xy + 8y^2$ and the constraint equation is $g(x, y) = 1$. Taking gradients yields the Lagrange multiplier system:

$$\begin{aligned} 3 &= \lambda(10x - 4y) \\ 1 &= \lambda(-4x + 6y) \\ 5x^2 - 4xy + 8y^2 &= 1. \end{aligned}$$

In the first equation, solve for λ to obtain $\lambda = 3/(10x - 4y)$. Substitute this into the second equation and cross multiply to obtain $10x - 4y = 3(-4x + 6y)$. Simplifying the equation shows that $x = y$ solves the equation. Substitute for y in the constraint equation and simplify to get $9x^2 = 1$. Thus we obtain two points, $\pm(\frac{1}{3}, \frac{1}{3})$. The maximum occurs at $(\frac{1}{3}, \frac{1}{3})$ and is equal to $\frac{1}{3}$. The minimum occurs at $-(\frac{1}{3}, \frac{1}{3})$ and is equal to $-\frac{7}{3}$.

2. Let $S = \{(x,y) : 0 \le y \le 2x \text{ and } 0 \le x \le 2\}$. Use a double integral to find the volume of the region of space lying between the graphs of the functions $g(x,y) = -y^2 - 1$ and $h(x,y) = x^2 + 1$ for $(x,y) \in S$. (*Hint:* Which graph lies above the other?)

h is larger than g on the domain, so we want to evaluate:

$$\begin{aligned}
\int_0^2 \int_0^{2x} h(x,y) - g(x,y)\, dydx &= \int_0^2 \int_0^{2x} x^2 + 1 - (-y^2 - 1)\, dydx \\
&= \int_0^2 \int_0^{2x} x^2 + y^2 + 2\, dydx \\
&= \int_0^2 \left[x^2 y + \frac{1}{3} y^3 + 2y \right]_0^{2x} dx \\
&= \int_0^2 \frac{14}{3} x^3 + 4x\, dx \\
&= \left[\frac{7}{6} x^4 + 2x^2 \right]_0^2 \\
&= \frac{7}{6} 2^4 + 2 \cdot 2^2 \\
&= \frac{80}{3}.
\end{aligned}$$