MATH 241

Sample First Hour Exam

February 10, 2008

Do your calculations in the blue book. Be sure to show your algebra. Each problem is worth 20 points.

1. **Short Answer.**

   (a) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^3$.

   i. What is the geometric formula for the length of the cross product of two vectors, that is, for $\| \mathbf{u} \times \mathbf{v} \|$?

   ii. If both $\| \mathbf{u} \|$ and $\| \mathbf{v} \|$ are held constant but their directions are allowed to vary, based on your answer to (i), when will $\| \mathbf{u} \times \mathbf{v} \|$ be largest?

   (b) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^3$.

   i. What is the geometric formula for the absolute value of the dot product of two vectors, that is, for $|\mathbf{u} \cdot \mathbf{v}|$?

   ii. If both $\| \mathbf{u} \|$ and $\| \mathbf{v} \|$ are held constant but their directions are allowed to vary, based on your answer to (i), when will $|\mathbf{u} \cdot \mathbf{v}|$ be largest?

2. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbf{R}^3$ and let a and b be scalars. Prove that

$$(a\mathbf{u} + b\mathbf{v}) \times \mathbf{w} = (a\mathbf{u} \times \mathbf{w}) + (b\mathbf{v} \times \mathbf{w}).$$

3. Let $\mathcal{P}$ be the plane passing through the three points $P_1 = (1, 1, 1)$, $Q_1 = (1, 1, -1)$ and $R_1 = (-1, -1, -2)$ and let $\mathcal{L}$ be the line passing through $P_2 = (3, 1, 0)$ and $Q_2 = (-2, -1, 1)$.

   (a) Find the coordinate equation for $\mathcal{P}$.

   (b) Find the vector equation for the $\mathcal{L}$.

   The equation is $P_2 + t(Q_2 - P_2) = (3, 1, 0) + t(-5, -2, 1)$.

   (c) Use your answers to (a) and (b) to find the point of intersection of $\mathcal{P}$ and $\mathcal{L}$.

4. Let $t = f(s)$ be a differentiable function and let $\boldsymbol{\alpha}(t) = (x(t), y(t))$ be a differentiable parametrization. Prove the chain rule for the derivative of the composition $(\boldsymbol{\alpha} \circ f)(s)$:

$$(\boldsymbol{\alpha} \circ f)'(s) = \boldsymbol{\alpha}'(f(s))f'(s).$$

5. Let $\mathcal{C}$ be the parametrization $\boldsymbol{\alpha}$ defined by

$$\boldsymbol{\alpha}(t) = (\cos(t), \cos(t)\sin(t))$$

for $0 \leq t \leq 2\pi$. (See the accompanying plot.)

(a) Based on the formula, place arrows on the plot to indicate which way the curve is traced.

(b) If we imagine the curve being traced by an object whose position is given by $\boldsymbol{\alpha}(t)$, for which values of t in the interval $0 \leq t \leq 2\pi$ will the object be at the origin?

(c) For each of the values you found in (b), compute the acceleration of the object.

(d) What does your answer to (c) tell you about the velocity at these points?

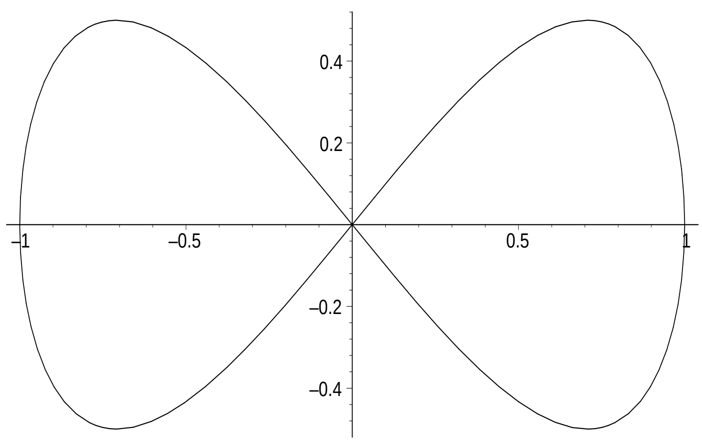

Figure 1: