

MATH 241
Second Hour Exam Sample
March 28, 2008

Do your calculations in the blue book. Be sure to show your algebra. Partial credit will be given.

1. (20 pts.) Suppose that $z = f(x, y)$ is defined on the entire plane.

(a) What does it mean to say that $f(x, y)$ is **continuous** at (x_0, y_0) ?

It means that $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$.

(b) Give an example of a function that is defined for all (x, y) , continuous for all $(x, y) \neq (0, 0)$, but not continuous at $(0, 0)$.

Let f be defined by $f(x, y) = x^2 + y^2$ for $(x, y) \neq (0, 0)$ and define $f(0, 0) = 1$. The function is continuous away from the origin because it is a polynomial. Since $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$ but $f(0, 0) = 1$, the function is not continuous at the origin.

(c) What does it mean to say that $f(x, y)$ is **differentiable** at (x_0, y_0) ?

It means that there is a linear function $l(x, y)$ with the property that

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x, y) - l(x, y)}{\| (x, y) - (x_0, y_0) \|} = 0.$$

(d) Give an example of a function that is continuous for all (x, y) , differentiable for all $(x, y) \neq (0, 0)$, but not differentiable at $(0, 0)$.

Let f be defined by $f(x, y) = \sqrt{x^2 + y^2}$ for all (x, y) . The partial derivatives are defined and continuous away from the origin, so it is differentiable away from the origin. However, the partial derivatives are not defined at the origin, so it is not differentiable there.

2. (20 pts.) Suppose that $z = f(x, y)$ is a twice differentiable function for all $(x, y) \in \mathbf{R}^2$. Let (x_0, y_0) be a local maximum for f .

(a) Define what it means for the maximum at (x_0, y_0) to be **isolated**.

A maximum is isolated if there is a disk of some radius $r > 0$ centered at (x_0, y_0) so that (x_0, y_0) is the only critical point of f in this disk.

(b) Define what it means for the maximum at (x_0, y_0) to be **non-degenerate**.

A maximum is non-degenerate if the second derivatives of f at (x_0, y_0) satisfy $AC - B^2 \neq 0$, where $A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$, $B = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$, and $C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$.

(c) Which of the following statements is false? (Give an example to illustrate your answer.)

i. If a local maximum is isolated, then it is non-degenerate.

False. Consider $f(x, y) = -x^4 - y^4$. It has an isolated global maximum (which is stronger than local) at $(0, 0)$, but $A = B = C = 0$.

ii. If a local maximum is non-degenerate, then it is isolated.

True. If it is non-degenerate the function behaves like its quadratic approximation near the point and the local maximum of a quadratic function is isolated.

3. (20 pts.) Let $f(x, y, z) = e^{-x^2-y^2-z^2}$ and $\alpha(t) = (t, t^2, t^3)$, and define $z = g(t)$ by $g(t) = (f \circ \alpha)(t)$.

(a) Use the chain rule to compute $g'(t)$.

$$\nabla f(x, y, z) = -2e^{-x^2-y^2-z^2}(x, y, z) \text{ and } \alpha'(t) = (1, 2t, 3t^2), \text{ so } g'(t) = -2e^{-t^2-t^4-t^6}(t, t^2, t^3) \cdot (1, 2t, 3t^2) = -2e^{-t^2-t^4-t^6}(t+2t^3+3t^5) - 2te^{-t^2-t^4-t^6}(1+2t^2+3t^4).$$

(b) Use your answer to (a) to find the intervals (in t) for which $g(t)$ is an increasing function and the intervals for which it is a decreasing function.

The calculation in (a) implies that g is increasing when $t < 0$ and decreasing when $t > 0$.

(c) What does your answer to (b) tell you about the critical points of g ?

The only critical point of g occurs when $t = 0$. Part (b) tells us that it must be a maximum.

4. (20 pts.) Let $f(x, y) = \frac{1}{3}(x - y)^3 - (x - y) + x^2$.

(a) Find the critical points of f . (*Hints:* (1) There are two critical points. (2) Don't expand expressions too much when solving for the critical points.)

$\nabla f(x, y) = ((x - y)^2 - 1 + 2x, -(x - y)^2 + 1)$. Setting this equal to 0, we see that the second term is zero when $(x - y)^2 = 1$. Substituting this into the first coordinate, we see that $1 - 1 + 2x = 0$, so $x = 0$. If $x = 0$, then $(x - y)^2 = 1$ says that $y^2 = 1$. Consequently, $y = \pm 1$. The critical points are $(0, 1)$ and $(0, -1)$.

(b) Use the second derivative test to identify the types of the critical points.

The second derivatives are: $\frac{\partial^2 f}{\partial x^2}(x, y) = 2(x - y) + 2$, $\frac{\partial^2 f}{\partial x \partial y}(x, y) = -2(x - y)$, and $C = \frac{\partial^2 f}{\partial y^2}(x, y) = 2(x - y)$. At $(0, 1)$, $A = 0$, $B = 2$, $C = -2$, so that $AC - B^2 = -4$. Therefore $(0, 1)$ is a saddle. At $(0, -1)$, $A = 4$, $B = -2$, $C = 2$, so that $AC - B^2 = 4$. Since $A > 0$ and $AC - B^2 > 0$, $(0, -1)$ is a minimum.

5. (20 pts.) Below is the plot for the vector field:

$$\mathbf{F}(x, y) = (3xy^2 - 7x, x^2y - 6y)$$

- (a) Use algebra to find the coordinates of the critical points of \mathbf{F} . (*Hint:* There are five critical points.)

Setting the coordinates equal to zero we have $0 = 3xy^2 - 7x = x(3y^2 - 7)$ and $0 = x^2y - 6y = (x^2 - 6)y$. The first equation implies that $x = 0$, or $y = \pm\sqrt{7/3}$. If $x = 0$, the second equation implies that $y = 0$. If $y = \pm\sqrt{7/3}$, the second equation implies that $x = \pm\sqrt{6}$. Therefore the critical points are at $(0, 0)$, $(\sqrt{7/3}, \sqrt{6})$, $(-\sqrt{7/3}, \sqrt{6})$, $(\sqrt{7/3}, -\sqrt{6})$, and $(-\sqrt{7/3}, -\sqrt{6})$.

- (b) Based on the plot, what is the type of each of the critical points you found in (a).

The origin is a sink and all the others are saddles.

- (c) Based on your answer to (b), is it possible or impossible that \mathbf{F} is a gradient vector field? Explain.

Yes is possible. None of the critical points are centers. Gradient vector fields have no centers, so this could be a gradient vector field.

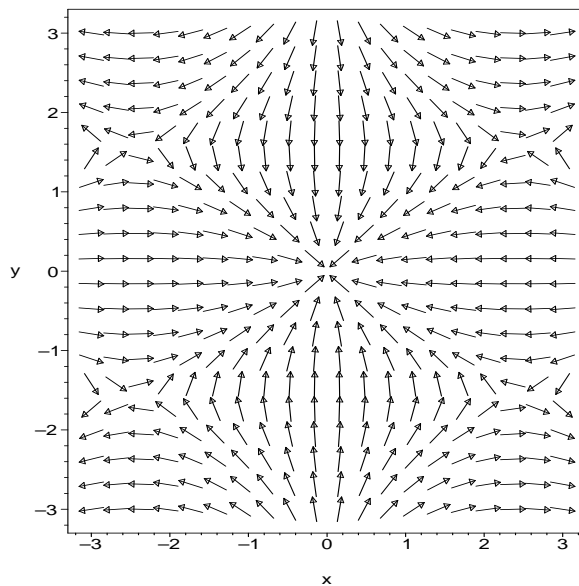


Figure 1: Vector field plot for problem 5.