Topics in Mathematics: Math and Music

The Monochord Lab: Length Versus Pitch

Due: Thursday, March 22, start of class

Names: ____

The goal of this lab project is for you to explore the relationship between the length of a string and the pitch sounded when the string is plucked. The device you will use to investigate this relationship is called a **monochord**, a simple one-stringed instrument that was commonly used in European classrooms during the Middle Ages and Renaissance. You will recreate a famous and ancient musical scale known as the **Pythagorean scale**. Once you have completed the first portion of the lab, I will check your work before passing out the second part of the lab.

You should work in a group of three to four people (one monochord per group) answering questions and filling in your results as you complete the lab. Please turn in one lab report per group, listing the names of all the members at the top of the first page. After everyone has completed Table 1, we will perform some simple pieces of music as a class. Before beginning the lab, please read the safety instructions below.

Safety

- The end of the string is sharp. Be careful when handling the monochord near the tuner end of the instrument.
- Over-tightening the string may cause it to break, allowing the two broken pieces to fly around somewhat. The broken ends will be sharp, so handle any broken string carefully. Keep your face away from the string so that you do not get hit in the eye by a breaking string. I have some spare strings if your string happens to break.
- This instrument is not for "twanging" as rock guitarists play their instruments. This type of playing may generate a louder sound, but the string life is reduced significantly. Sufficient volume may be generated with just a gentle "pluck" of the string.

The Pythagorean Scale

According to legend, the great Greek mathematician and philosopher Pythagoras was walking by a blacksmith's shop when he noticed that certain sounds of the hammers on the anvils were more consonant than others. Upon investigation, he discovered that the nicer sounds came from hammers whose weights were in simple proportions like 2 : 1, 3 : 2, and 4 : 3. After further musical experiments with strings and lutes, he and his followers (the Pythagoreans) became convinced that basic musical harmony could be expressed through simple ratios of whole numbers. Their general belief was that the lower the numbers in the ratio, the better the notes sounded together. Thus, the natural building blocks of mathematics (small whole numbers) were aligned with the natural interval relationships used to create harmonious music. This helps explain the devotion of the Pythagoreans to rational numbers, numbers that can be expressed as the ratio of two integers.

Using the monochord, you will investigate this Pythagorean belief and find the ratios used in the oldest musical scale, the Pythagorean scale. The goal is to investigate the relationship between the length of the string vibrating and the pitch produced.

Part I: Simple Ratios

1. The tension of the string may be adjusted by tightening the peg at one end of the monochord. What happens to the pitch as the string is tightened?

2. Place the sliding fret exactly at 30 cm. Hold one finger over the fret and pluck the string gently. What is the relationship between the pitch of this note versus the note produced without the fret? If you are not sure about the relationship, you can try finding the notes on the piano, although first you will have to "tune" the monochord (on the open, unfretted string) to a note on the piano.

3. Place the sliding fret exactly at 15 cm. Hold the string down over the fret and pluck the smaller portion of the string (so you are vibrating a string of length 15 cm.) What is the musical relationship (e.g., interval) between this tone and the one obtained by plucking a string of length 30 cm? What is the relationship between the 15 cm pitch and the open plucked string (60 cm)?

4. Place the sliding fret exactly at 20 cm, hold the string down over the fret and pluck each string (20 cm and 40 cm). What is the musical interval between these two pitches? Draw an important conclusion:

By cutting the length of the string in **half**, we ____________ (raise or lower) the pitch by _________________ (what musical interval)?

5. Next, investigate what happens if the ratio of the string lengths is 2 : 3. In other words, what effect does changing the string length to 2/3 its original value have on the pitch? Be sure to pluck the correct side of the string. *Hint:* What is 2/3 of 60?

6. Next, investigate what happens if the ratio of the string lengths is 3 : 4. In other words, what effect does changing the string length to 3/4 its original value have on the pitch? Again, be sure to pluck the correct side of the string. Given that $\frac{3}{4} = \frac{1}{2} \cdot \frac{3}{2}$, how could you have predicted this result from the answers to the previous two questions?

Part 2: Ratios for the Pythagorean Scale

Given the ground work accomplished above, we are now ready to construct the entire Pythagorean scale. Although it sounds like the usual major scale, it differs in some important ways from the scale found on a modern piano. These differences will be explored in great detail when we study tuning and temperament (Chapter 4).

Thus far, we have uncovered three important facts concerning the relationship between the ratios of string lengths and the corresponding musical interval.

1. String lengths in a ratio of $1:2$ are an octave apart. Specifically, cutting the length of the string in half raises the pitch an octave. Conversely, doubling the length of the string lowers the pitch an octave.

2. String lengths in a ratio of $2:3$ are a perfect fifth apart. Specifically, cutting the length of the string by a factor of $2/3$ raises the pitch a perfect fifth.

3. String lengths in a ratio of $3:4$ are a perfect fourth apart. Specifically, cutting the length of the string by a factor of $3/4$ raises the pitch a perfect fourth.

Notice the musical and mathematical simplicity of these facts. Simple ratios lead to the "perfect" intervals of an octave, fifth, and fourth. It is no accident that these are the primary intervals of early music (e.g., Gregorian chant used octaves, while Medieval polyphony used fourths and fifths.) They also underscore the critical tonic-dominant harmonic relationship and the V–I and I–IV–V chord progressions commonly found in music.

It is also important to note that fact 3 follows from the first two facts. Suppose that the open string at 60 cm sounds middle C on the piano. Then by fact 1, a string of length 30 cm will sound the C an octave higher than middle C, call it C'. By fact 2, a string of length 40 cm will sound the G above middle C (a perfect fifth higher.) Note that the interval between G and C' (G being the lower note) is a perfect fourth. Thus, since the ratio between the two strings sounding G and C' is $30:40$ or $3:4$, the net effect of shortening the string length by a factor of $3/4$ is to raise the pitch by a perfect fourth. In other words, since $\frac{3}{4} = \frac{1}{2} \cdot \frac{3}{2}$, the ratio for going up an octave and down by a perfect fifth (which is equivalent to going up a perfect fourth) is $3:4$.

The goal for part 2 of the lab is to discover the remaining ratios in the Pythagorean scale. The scale is essentially the same as the major scale except the string lengths are slightly different from the modern ones. We will derive this scale using only facts 1 and 2 above and the circle of fifths. For example, to find the second note of the major scale, we go up two perfect fifths from the tonic and then down an octave. The net effect raises the starting pitch by a whole step. If the starting note is middle C, then going up by a perfect fifth gives G, and going up another perfect fifth gives D'. We then go down an octave to obtain the D just above middle C. Using the ratios given in facts 1 and 2, what fraction of the full string do we take to obtain the second note of the scale? Use a calculator to find the actual length of the string (round to two decimal places) and play it with the open string to hear the first two notes of the Pythagorean scale.

The next note to find would be the sixth scale degree since that is a perfect fifth above the second. From there we can find the third and finally the seventh note of the scale, the leading tone. Fill out the table below as you find the lengths and ratios of the string. When you have found the complete scale, play it and see if you can hear any differences with the scale on the modern piano.

Scale Degree	**Solfège Syllable**	**Interval**	**Ratio**	**Length in cm**
1	Do	Unison	$\frac{1}{1} \cdot 60$	60
2	Re	Major Second		
3	Mi	Major Third		
4	Fa	Perfect Fourth	$\frac{3}{4} \cdot 60$	45
5	Sol	Perfect Fifth	$\frac{2}{3} \cdot 60$	40
6	La	Major Sixth		
7	Ti	Major Seventh		
$8 = 1$	Do	Octave	$\frac{1}{2} \cdot 60$	30

Table 1: The lengths and ratios of the Pythagorean scale. Give lengths to two decimal places.

Some Concluding Questions

1. Now that you have completed the table, find the prime factorization of the numerator and denominator for each ratio in the fourth column of your table. For example, $3/4 = 3/2^2$. Generally speaking, what do you notice about each factorization?

2. Using your table, what is the ratio of string lengths that are a whole step apart? You should check **all** five whole steps in the major scale (show your work) to see that the ratio is identical for each whole step. This ratio, call it W, is the factor you would multiply the length by to raise the pitch a whole step. Express W as a ratio of two integers.

3. Using your table, what is the ratio of string lengths that are a half step apart? You should check **both** half steps in the major scale (show your work). This ratio, call it H, is the factor you would multiply the length by to raise the pitch a half step. Express H as a ratio of two integers (you might choose to use the prime factorization of the numerator and denominator.)

4. Since two half steps is equivalent to a whole step, it should be true that $H^2 = W$. Note that H is squared here because we must *multiply* twice to raise the pitch two half steps. Compute the value of H^2/W, giving your answer as a ratio of two integers and in decimal form (use 5 decimal places). The fact that this value is not equal to one is a problem! If two half steps do not equal one whole step, how can we expect to play in a different key? If we have a melody that moves from C to C$^\sharp$ to D (two half steps), then we obtain a different "D" from the one found by moving from C to D (one whole step). This is one of the major drawbacks of the Pythagorean scale. The value of H^2/W is called the **Pythagorean Comma**.