Topics in Mathematics: Mathematics and Music
Section 1.1: Musical Notation and a Geometric Property
January 25, 2018

1 Duration: Geometric Sequences

Figure 1: Some basic symbols and terminology used for musical notation. In the rightmost figure, the line connecting the two notes is called a *beam*.

The number or fraction of beats a note is held depends on whether the note head is shaded or not, if it has a stem, and if there are any flags. There is also the possibility of adding a dot to a note to increase its length. Table 1 shows the different types of notes, their names, and their durations. Here we assume that a quarter note has a length that represents one beat.

Symbol:	𝅝	𝅗𝅥	♩	♪	𝅘𝅥𝅯	𝅘𝅥𝅰
Name of note:	whole	half	quarter	eighth	sixteenth	thirty-second
Number of beats:	4	2	1	1/2	1/4	1/8

Table 1: The different types of notes and their durations, assuming that a quarter note equals one beat.

Notice the interesting mathematical pattern in the durations of the notes in Table 1. Each successive note is half the length of the previous note. The list of durations of the different notes,

$$4,\ 2,\ 1,\ 1/2,\ 1/4,\ 1/8,\ 1/16,\ \ldots,$$

is a nice example of a *geometric sequence* with ratio $r = 1/2$.

Definition 1.1 *A geometric sequence is a list of numbers where each successive number is obtained by multiplying the previous number by a common nonzero ratio r. If a_n represents the nth term in the sequence, then the next term is given by $a_{n+1} = r \cdot a_n$.*

Example 1.2 *The sequence* $1,\ 3,\ 9,\ 27,\ 81,\ \ldots$ *is a geometric sequence with common ratio* $r = 3$. *Each term, except for the first, is the geometric mean of its adjacent neighbors. For example,* $3 = \sqrt{1 \cdot 9}$ *and* $9 = \sqrt{3 \cdot 27}$. *The sequence* $6,\ -4,\ 8/3,\ -16/9,\ \ldots$ *is also a geometric sequence, but here the common ratio is negative, with* $r = -2/3$. *The sequence* $2,\ 4,\ 6,\ 8,\ 10,\ \ldots$ *of even numbers is arithmetic, not geometric. The sequence* $1,\ 2,\ 4,\ 7,\ 13,\ 24,\ \ldots$ *is not geometric because there is no common ratio that works between all pairs of successive numbers in the list (e.g.,* $4/2 \neq 7/4$*).*
□

Exercise 1

(a) Write out the first six terms of a geometric sequence that starts with first term $a_0 = 64$ and common ratio $r = 1/4$.

(b) Determine whether the sequence $27, -18, 12, -8, 16/3, \ldots$ is a geometric sequence or not. If so, state the common ratio r. What are the next two terms in the sequence?

A piece of music is typically divided up into *measures*, where each measure is separated by a vertical bar called a *bar line*. The number of beats per measure can vary between works, or can even change multiple times during a piece. This is notated using a *time signature*, an important musical concept we discuss in the next section. The most commonly used time signature is $\frac{4}{4}$, called *common time* and denoted by **c**, which has four beats per measure with the quarter note receiving one beat. A simple excerpt of music in common time is shown in Figure 2. The excerpt contains six measures, each with exactly four beats of music. The location of the notes on the staff is not important at this point; focus on the type of note (sixteenth, eighth, quarter, half, or whole), and the fact that each measure contains a total of four beats.

Figure 2: A simple excerpt of music in $\frac{4}{4}$ time containing six measures, each of which has four beats with the quarter note equal to one beat. A single beam connecting two stems produces two eighth notes (e.g., measures 2 and 6), while a double beam between stems creates sixteenth notes (e.g., measure 5). **Exercise 2:** In each measure, place the numbers 1, 2, 3, 4 over the notes to indicate the location of each beat.

In Table 2, for a particular type of note, we list the number that is needed to fill one measure in common time. We also list what fraction of the measure is filled by a given note. Not surprisingly, two geometric sequences are apparent. The third row is a geometric sequence with common ratio $r = 2$, while the fourth row is another sequence with $r = 1/2$. Notice that the terms in the second row correspond exactly with the fractions in the fourth row, revealing the meaning behind the different names of the notes.

As mentioned above, there is also notation for silence, denoted by *rests*. The notation to indicate that a musician should rest for a specific number or fraction of beats is indicated in Table 3. Notice the similarities between the structure and names of the rests in Table 3 and the notes in Table 1.

Symbol:	𝅝	𝅗𝅥	♩	♪	𝅘𝅥𝅯	𝅘𝅥𝅰
Name of note:	whole	half	quarter	eighth	sixteenth	thirty-second
Number required to fill a four-beat measure:	1	2	4	8	16	32
Fraction of time held in a four-beat measure:	1	1/2	1/4	1/8	1/16	1/32

Table 2: The different types of notes and their durations in terms of a four-beat measure where the quarter note equals one beat.

Symbol:	𝄻	𝄼	𝄽	𝄾	𝄿	𝅀
Name of rest:	whole	half	quarter	eighth	sixteenth	thirty-second
Number of beats:	4	2	1	1/2	1/4	1/8

Table 3: The different types of rests and their durations, assuming that a quarter note equals one beat. A whole rest and half rest are each denoted by a small solid rectangle, except that a whole rest lies directly *below* the fourth line of the staff, while a half rest lies directly *above* the third line of the staff.

2 Dots: Geometric series

One common technique for extending the duration of a note is to add a dot after it. This has the effect of increasing the length the note is held by one-half its original value. For example, a dotted half note in $\frac{4}{4}$ time will get two beats for the half note plus one extra beat for the dot, giving a total of three beats. We can also have notes with two, three, or even four dots added. This illustrates the mathematical concept of *iteration*, where the same process is applied repeatedly. A note with two dots will be lengthened by half the original value and then lengthened again by half of the new value. Each additional dot lengthens the note by half the value of the previous dot. Thus, a double-dotted half note in $\frac{4}{4}$ time will get two beats for the half note, one beat for the first dot, and an additional half a beat for the second dot, yielding a total of $3\frac{1}{2}$ beats. It is also possible to add dots to rests to increase their length. The same rule applies: each dot increases the length of the rest by one-half its original value. Table 4 demonstrates the result of adding more and more dots to increase the duration of a half note in $\frac{4}{4}$ time.

Exercise 3:

Suppose that the length of a quarter note corresponds to one beat. How many beats does a triple-dotted quarter note get? How many beats does a double-dotted eighth note get?

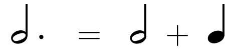
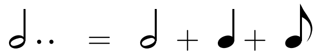
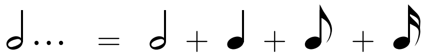
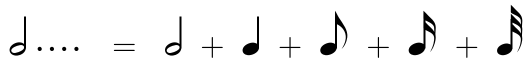
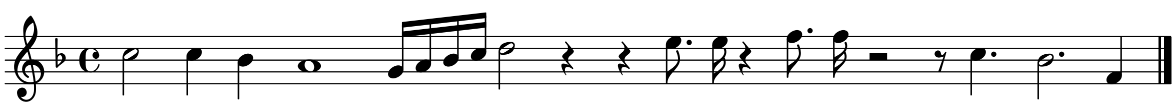
Note	Total number of beats
	2
	$2 + 1 = 3$
	$2 + 1 + \frac{1}{2} = 3\frac{1}{2}$
	$2 + 1 + \frac{1}{2} + \frac{1}{4} = 3\frac{3}{4}$
	$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 3\frac{7}{8}$

Table 4: The effect of repeatedly adding dots to a half note. The number of beats is given assuming that a quarter note equals one beat.

Exercise 4: The following excerpt is in common time, which means that the quarter note is equal to one beat and there are four beats in each measure. However, the bar lines, indicating where each measure ends, are missing. Insert the bar lines correctly into the excerpt.



From a mathematical perspective, each sum in the right-hand column of Table 4 is an example of a *geometric series*. A geometric series is just a sum of the terms in a geometric sequence. In general, a series is a sum, while a sequence is a list. For example, suppose that we have a geometric sequence with a common ratio $r = 2$, a first term of $1/4$, and seven terms. Then, the corresponding geometric series is

$$\frac{1}{4} + \frac{1}{2} + 1 + 2 + 4 + 8 + 16 \; = \; 31\tfrac{3}{4}. \tag{1}$$

Definition 2.1 *A* <u>*geometric series*</u> *is a sum of the terms in a geometric sequence. It has the form*

$$S_n \; = \; a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^{n-1}, \tag{2}$$

where r is the common ratio and a_0 denotes the first term.

By examining the exponents, we note that there are n terms being summed in the series in Equation (2).

An explicit formula for the sum of the first n terms in a geometric series with ratio r and first term a_0 is

$$\boxed{S_n \; = \; \frac{a_0 - a_0 r^n}{1 - r}.} \tag{3}$$

We will derive this formula in class.

Exercise 5: Using the formula in equation (3), check the example in equation (1) to see that the formula gives the correct value.

As a slight but fun mathematical detour, let us consider what happens if we sum all of the terms in an *infinite* geometric series. While this may not have any specific musical applications, working with infinity (∞) is a common mathematical endeavor.

At first, summing an infinite geometric series may seem like an impossible task. After all, how can we sum an infinite set of numbers if we never stop adding? Formula (3) provides a way around this apparent conundrum. The key observation is that if the common ratio r satisfies $|r| < 1$, then the quantity r^n becomes smaller and smaller (in absolute value) as n gets larger and larger. For instance, if $r = 1/2$, then the sequence r^n is the familiar

$$1/2,\ 1/4,\ 1/8,\ 1/16,\ 1/32,\ 1/64,\ 1/128,\ 1/256,\ \ldots,$$

a sequence of values getting closer and closer to zero.

The relevant mathematical concept here is called a *limit*, abbreviated lim. Ignoring the rigorous mathematical details, we have

$$\lim_{n\to\infty} r^n \;=\; 0 \qquad \text{provided that } |r| < 1.$$

This implies that the quantity $a_0 r^n$, found in the numerator of Formula (3), will vanish "in the limit." This yields a simple formula for the sum of an infinite geometric series with common ratio r and first term a_0:

$$\boxed{S_\infty = \frac{a_0}{1-r}.} \tag{4}$$

It should be stressed that Formula (4) is valid *only* when the absolute value of the ratio r is less than 1. If this is not the case, then we say the infinite series *diverges* (does not limit on a particular numerical value).

Exercise 6: Find the sum of the given infinite geometric series.

a. $3 + 1 + 1/3 + 1/9 + \cdots$

b. $3 - 1 + 1/3 - 1/9 + - \cdots$