# Topics in Mathematics: Math and Music
## Comparing the Three Tuning Systems

| **Scale Degree** | **Solfège** | **Interval** | **Pythagorean** | **Just Intonation** | **Equal Temp.** |
|---|---|---|---|---|---|
| 1 | Do | Uni. | 1 | 1 | 1 |
| 2 | Re | M2 | $\frac{9}{8} = 1.125$ | $\frac{9}{8} = 1.125$ | $2^{2/12} \approx 1.1225$ |
| 3 | Mi | M3 | $\frac{81}{64} = 1.265625$ | $\frac{5}{4} = 1.25$ | $2^{4/12} \approx 1.2599$ |
| 4 | Fa | P4 | $\frac{4}{3} = 1.3\bar{3}$ | $\frac{4}{3} = 1.3\bar{3}$ | $2^{5/12} \approx 1.3348$ |
| 5 | Sol | P5 | $\frac{3}{2} = 1.5$ | $\frac{3}{2} = 1.5$ | $2^{7/12} \approx 1.4983$ |
| 6 | La | M6 | $\frac{27}{16} = 1.6875$ | $\frac{5}{3} = 1.6\bar{6}$ | $2^{9/12} \approx 1.6818$ |
| 7 | Ti | M7 | $\frac{243}{128} = 1.8984375$ | $\frac{15}{8} = 1.875$ | $2^{11/12} \approx 1.8877$ |
| $8 = 1$ | Do | Oct. | 2 | 2 | 2 |

Table 1: The ratios or multipliers used to **raise** a note (increase the frequency) by a given musical interval in the three different tuning systems: the Pythagorean scale, just intonation, and equal temperament. Pythagorean tuning and just intonation use rational numbers while equal temperament uses irrational multipliers (except for unison or the 2:1 octave).

**Example 0.1** *Find the frequency of the note $C^\sharp$ above A440 Hz in each of the three different tuning systems.*

**Example 0.2** *Find the frequency of the note F just below A440 Hz in each of the three different tuning systems.*

| Scale Degree | Solfège | Interval | Pythagorean | Just Intonation | Equal Temp. |
|---|---|---|---|---|---|
| 1 | Do | Uni. | 0 | 0 | 0 |
| 2 | Re | M2 | 203.9 | 203.9 | 200 |
| 3 | Mi | M3 | 407.8 | 386.3 | 400 |
| 4 | Fa | P4 | 498.0 | 498.0 | 500 |
| 5 | Sol | P5 | 702.0 | 702.0 | 700 |
| 6 | La | M6 | 905.9 | 884.4 | 900 |
| 7 | Ti | M7 | 1109.8 | 1088.3 | 1100 |
| $8 = 1$ | Do | Oct. | 1200 | 1200 | 1200 |

Table 2: A comparison of the three tuning systems using **cents**, rounded to one decimal place. Note that equal temperament does a good job approximating a perfect fifth (only 2 cents off), but is noticeably sharp (nearly 14 cents) of a just major third.

**Cents** were introduced by the mathematician Alexander Ellis (1804–90) around 1880. They are now a commonly used unit of measurement when comparing different tuning systems, or discussing non-traditional tunings. A typical listener can distinguish pitches that are between 4 and 8 cents apart. Cents are based on a logarithmic scale (like decibels). The formula for converting a ratio or multiplier $r$ into cents is

$$\boxed{\#\text{ of cents} \;=\; 1200\log_2(r) \;=\; 1200 \cdot \frac{\ln r}{\ln 2}\,.} \tag{1}$$

For example, a half step in equal temperament is given by the multiplier $2^{1/12}$. In cents, using the definition of the logarithm, this is

$$1200\log_2(2^{1/12}) \;=\; 1200 \cdot \frac{1}{12} \;=\; 100 \text{ cents.}$$

Since all half steps are equal in equal temperament, one can easily obtain any interval in cents just by multiplying the number of half steps in the interval by 100 (see Table 2). For comparison, the Pythagorean comma is approximately 23.5 cents while the syntonic comma is roughly 21.5 cents.

**Example 0.3** *Compute the number of cents in a minor third in each of the three different tuning systems.* **Note:** *The ratio to raise the pitch a minor third in Pythagorean tuning is* $32/27$ *(Exercise 6 in Section 4.1).*