Topics in Mathematics: Math and Music

Section 3.3: Sine Waves

Radians

Angles in trigonometry are measured in radians, which corresponds to an actual physical length (as opposed to degrees, which is based on the fact that 360 has many factors).

Definition 0.1 *An angle of 1 radian is equal to the angle made by 1 unit of arc length along the unit circle.*

To convert between radians and degrees, use the formula

$$\boxed{180^\circ \;=\; \pi \text{ radians.}} \tag{1}$$

Exercise 0.2 *Convert each of the following from degrees to radians:*

a. $90^\circ \;=$ ______ **b.** $270^\circ \;=$ ______ **c.** $60^\circ \;=$ ______ **d.** $-450^\circ \;=$ ______

The Sine and Cosine Functions

There are multiple definitions of the sine function $y = \sin t$, but one of the simplest to understand is that it represents the y-coordinate on the unit circle at an angle of t radians.

Definition 0.3 *The sine of t, denoted by $\sin(t)$ or just $\sin t$, is the y-coordinate of the point of intersection between the unit circle and a ray emanating from the origin at an angle of t radians. The cosine of t, denoted by $\cos(t)$ or just $\cos t$, is the x-coordinate.*

It is important to remember that the input into the functions $f(t) = \sin t$ or $g(t) = \cos t$ is an *angle* t. Since the unit circle has a radius of one, the x- and y-coordinates of any point on the unit circle always lie between -1 and 1. Thus, the range (output) of $\sin t$ and $\cos t$ is $[-1, 1]$. By the Pythagorean Theorem, we have a fundamental relationship between $\sin t$ and $\cos t$:

$$\boxed{\cos^2(t) + \sin^2(t) \;=\; 1 \quad \text{for any angle } t.} \tag{2}$$

The notation $\cos^2(t)$ means take the cosine of t and then square the result, that is, $\cos^2(t) = (\cos(t))^2$.

Unit Circle

Since the cosine and sine functions are defined in terms of the coordinates on the unit circle, it is absolutely critical to have a solid understanding of the unit circle and its key values.

Exercise 0.4 *Without using a calculator, find the values of each expression:*

a. $\sin(0) =$ ______ **b.** $\sin(\pi) =$ ______ **c.** $\sin(12\pi) =$ ______

d. $\cos(13\pi) =$ ______ **e.** $\cos(9\pi/2) =$ ______ **f.** $\cos^2(\pi/75) + \sin^2(\pi/75) =$ ______

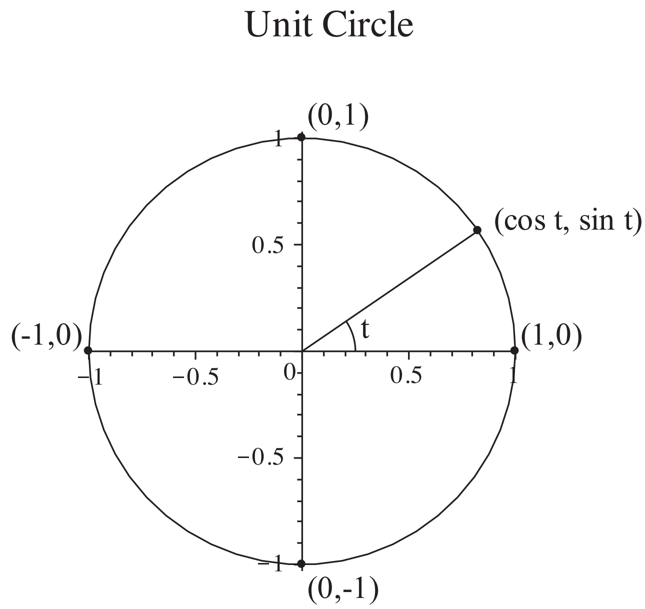
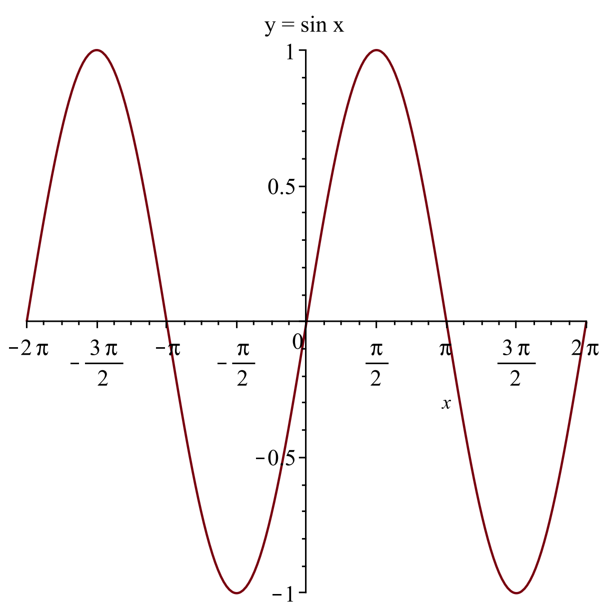


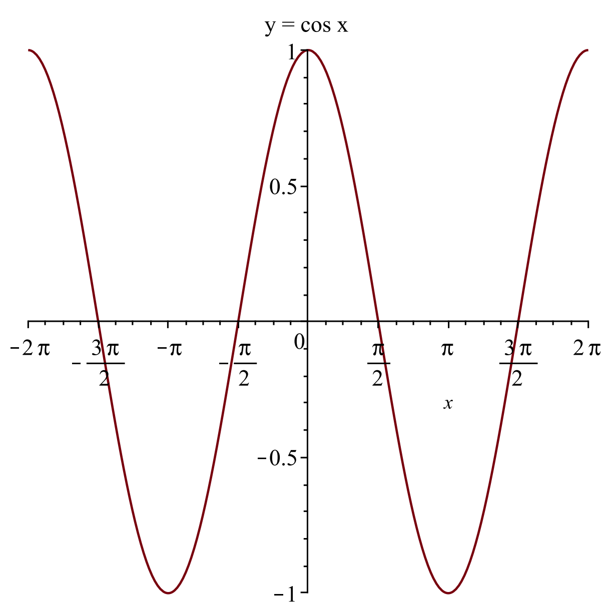
Figure 1: The cosine and sine functions are determined by the x- and y-coordinates, respectively, on the unit circle. You should **memorize this figure!**

Graphing sinusoids

Recall that when a tuning fork vibrates, it emits a sound wave in the form of sine curve (called a **sinusoidal wave**). Sine waves are the building blocks of musical sound. Thus, it is important to know how to graph the sine function and to understand how properties such as amplitude and period effect the graph.

Below are the graphs of $\sin x$ and $\cos x$ showing two full cycles. These are excellent examples of **periodic** functions, that is, functions that repeat themselves after some time, called the **period**. The period of $\sin x$ and $\cos x$ are each 2π because the values repeat themselves every full turn around the unit circle.





Period and Amplitude

The **period** of a function that retraces itself repeatedly is the length of time it takes between successive cycles. The period can be measured from the crest of the wave to the next crest, or from the trough of the wave to the next trough. To alter the period of a sine wave, we adjust the speed we are traveling around the unit circle (replace t by $b\,t$).

The height of the wave, called the **amplitude**, is defined to be the distance from the top of the wave to its horizontal axis of symmetry. This is the same as half the total vertical distance of the curve (from the highest to the lowest points of the wave). Hence, the amplitude of $y = \sin t$ is 1, which is half of 2, the distance between 1 and -1. To adjust the amplitude of a sine wave, we change the height of the graph (multiply by a in front).

The sine curve $y = a\sin(b\,t)$ has an amplitude of a and a period of $\dfrac{2\pi}{b}$. (3)

Note: In the formula above, a and b are assumed to be positive constants.

Exercise 0.5 *Find the amplitude and period of the function $y = 3\sin(100t)$. Sketch three cycles of the graph.*

Solution: Using Formula (3), we have $a = 3$ and $b = 100$, so the amplitude is 3 and the period is $2\pi/100 = \pi/50$. The graph of this sine function is shown below.

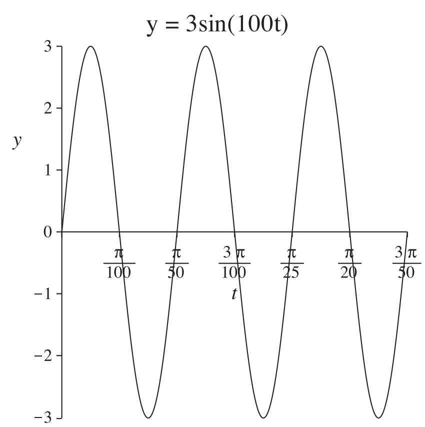


Figure 2: The graph of three cycles of $y = 3\sin(100t)$. The amplitude is 3 and the period is $\pi/50$.

Exercise 0.6 *Sketch a graph of the following trig functions. State the amplitude and period in each case.*

a. $y = 2\sin(4t)$

b. $y = 5\sin\left(\dfrac{\pi}{2}t\right)$

Phase Shift

The last important concept concerning sinusoids is the **phase shift**, which adjusts the starting point of the wave. Consider the effect of replacing t by $t - \pi/2$ in the standard sine function. The angle $t = 0$ shifts to become $-\pi/2$, while the angle $\pi/2$ shifts to 0. The angle $\pi/2$ has become the new starting point at $(1, 0)$. It follows that the usual sine wave will now be shifted to the *right* by $\pi/2$. Instead of beginning at $(t = 0, y = 0)$, the wave $y = \sin(t - \pi/2)$ begins at $(t = \pi/2, y = 0)$. The graph of this shifted wave is shown below. We say that the **phase shift** of the new wave is $\pi/2$. In general, if t is replaced by $t - c$, then the graph of the new function $f(t - c)$ will be a translation to the right of the old one by c units.

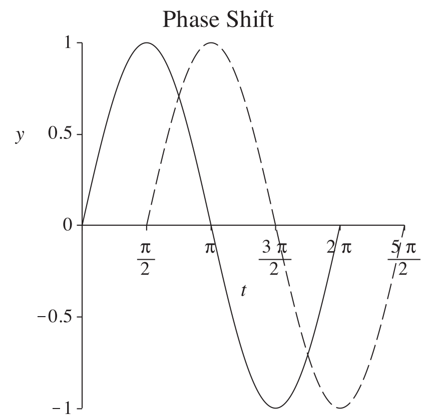


Figure 3: The graph of the usual sine wave $y = \sin t$ (*solid curve*) alongside its shifted version $y = \sin(t - \pi/2)$ (*dashed curve*).

Frequency and Period

Recall that the **frequency** of a wave is the number of cycles per second (measured in Hz). The pitch A440 vibrates with 440 cycles in one second. It follows that the period, the time it takes to do one cycle, is then $1/440$ (the reciprocal of the frequency).

$$\boxed{\text{frequency} \quad = \quad \frac{1}{\text{period}} \quad \text{or} \quad \text{period} \quad = \quad \frac{1}{\text{frequency}}.} \tag{4}$$

For a general sine wave $y = a \sin(b\,(t - c))$, a is the amplitude, $2\pi/b$ is the period, and c is the phase shift. The frequency is the reciprocal of the period, or $b/(2\pi)$.

Exercise 0.7 *The vibrations of a tuning fork cause the wave* $y = 75 \sin(440\pi(t - 0.01))$ *to propagate. Find the amplitude, period, phase shift, and frequency of the wave. What note is being sounded by the tuning fork?*