Topics in Mathematics: Math and Music
Worksheet for Section 5.3: Group Theory

Definition: The set G is a **group** under the operation $*$ if the following four properties are satisfied:

1. **Closure:** If $a \in G$ and $b \in G$, then $a * b \in G$. This must be true for all elements a and b in the group G.
2. **Associativity:** $(a * b) * c = a * (b * c)$ for any elements $a, b, c \in G$.
3. **Identity:** There must exist an element $e \in G$ called the **identity element** such that $a * e = a$ and $e * a = a$. (e preserves the "identity" of the element it is being multiplied by.)
4. **Inverse:** For every element $a \in G$, there must exist an element $a^{-1} \in G$ called the **inverse** of a, such that $a * a^{-1} = e$ and $a^{-1} * a = e$. Note that the inverse of each element must be in the group G.

Some Examples

- $G = \mathbb{Z}$ (the integers) with $* = +$, the usual addition of two integers. For example, $3 * 5 = 8$ because $3 * 5$ really means $3 + 5$. In this case, the identity element is $e = 0$ and $a^{-1} = -a$ since $a * a^{-1} = a + (-a) = 0 = e$. Note that a^{-1} is always in G because the negative of any integer is still an integer. Since addition is commutative, we have that $a * b = b * a$ for all a and b in the group. A group with this property is called a **commutative group**.

- $G = \mathbb{R} - \{0\}$ (all real numbers except for 0) with $* = \times$, the usual multiplication of two real numbers. In this group, $3 * 5 = 15$ because $3 * 5$ really means $3 \cdot 5$. Here, $e = 1$ and $a^{-1} = 1/a$. Why did we have to exclude 0 from G? This group is also an example of a commutative group.

  Note: The set $G = \mathbb{Z}$ (the integers) is **not** a group under multiplication. How come?

- $G = \{0, 1, 2, \ldots, 10, 11\}$ with $* = +$ mod 12 (modular arithmetic). In other words, when we add two elements together, if the result is greater than 11, we subtract a multiple of 12 until it is in G. For example,

  $$9 * 5 \;=\; 9 + 5 \;=\; 14 \;=\; 2 \pmod{12} \qquad \text{(since } 14 - 12 = 2\text{)},$$

  or

  $$8 * 4 \;=\; 8 + 4 \;=\; 12 \;=\; 0 \pmod{12} \qquad \text{(since } 12 - 12 = 0\text{)}.$$

  This commutative group is often denoted $\mathbb{Z}_{12}$ and is called the **integers mod 12**. Although this group may seem strange, it is exactly how we tell time in countries that use 12-hour clocks. For example, 2 hours after 11:00 is 13:00 in Britain, but 1:00 in the U.S., so 13:00 = 1:00 (mod 12). What is e for this group? What is the formula for a^{-1}?

  Musically speaking, this group is akin to identifying the same note in different octaves. If the various instruments in an orchestra are all playing C in different registers, we still think of the note heard as just a "C," even if the frequencies are different. Identifying all integers (or notes) that are multiples of 12 (or octaves) apart from each other, is called forming an **equivalence class**. This concept is a key guiding principle in **twelve-tone music**, discussed in Chapter 7 of the course text.

Multiplication Tables

One of the best ways to understand a group and its inherent structure is to create a **multiplication table**, a square table that shows all of the possible products from within the group. This only makes practical sense for groups with a finite number of elements, known as **finite groups**. The groups defined using modular arithmetic are examples of finite groups. A group multiplication table is sometimes called a **Cayley table**, named in honor of the English mathematician Arthur Cayley (1821–95).

Example: Complex Multiplication

Recall the complex number $i = \sqrt{-1}$, defined via the equation $i^2 = -1$. Let $G = \{1, i, -1, -i\}$ and $*$ correspond to complex multiplication. The multiplication table for G consists of all possible products of the elements of G (16 entries). The second row of the table below consists of the four products $1 * 1 = 1, 1 * i = i, 1 * -1 = -1$, and $1 * -i = -i$. Continuing in this fashion, complete the table below by multiplying each entry in the leftmost column by each entry in the top row. Then confirm that G is indeed a group by checking the four properties (you may assume that complex multiplication is associative.) What is the identity element e? What is the inverse of each element?

$*$	1	i	-1	$-i$
1	1	i	-1	$-i$
i				
-1				
$-i$				

Table 1: The multiplication table for $G = \{1, i, -1, -i\}$ under complex multiplication.

Example: Symmetries of the Square

Groups can arise in many different arenas, not just in the realm of real or complex numbers. One group that has some interesting connections to music involves the symmetries of a square. We start with a square with vertices labeled $1, 2, 3, 4$ in clockwise order. A **symmetry of the square** is a geometric transformation that preserves the size and shape of the square.

For example, rotating the square 90° clockwise about it's center returns the square to its original position, but with a different labeling of the vertices. We denote this rotation as R_{90}. Other possible rotations include R_{180} and R_{270}. A rotation of 360° returns the square to the same position with the same labeling, so this is really the identity element e (the transformation that doesn't move the square). These four symmetries are shown below.

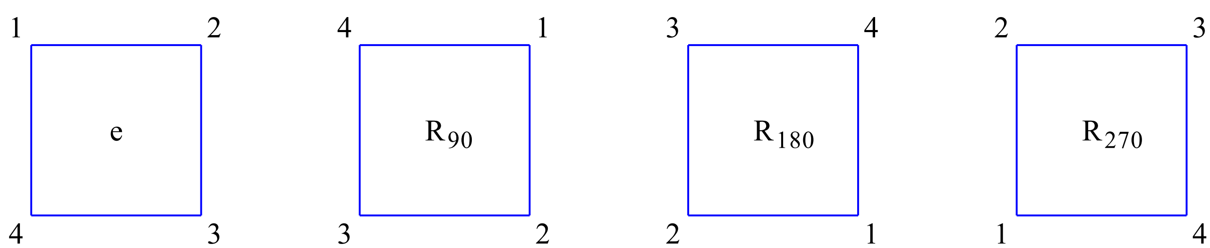

Figure 1: The original square along with rotations of $90, 180$, and 270 degrees.

In addition to rotations, it is possible to reflect the square about a line of symmetry and still preserve the original shape. There are four such lines: a horizontal line through the center (H), a vertical line through the center (V), a southeast diagonal between vertices 1 and 3 (D_{13}), and a northeast diagonal between vertices 2 and 4 (D_{24}). In Figure 2, we display the four reflections along with the corresponding labels on the vertices. These are the only possible reflections that preserve the original position of the square.

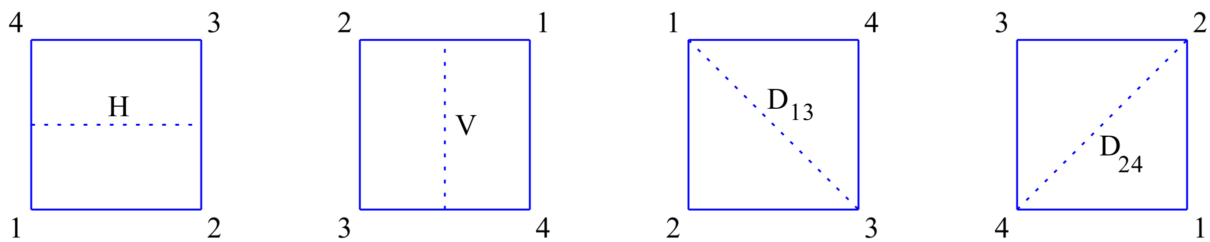

Figure 2: The four reflections of the original square through a horizontal, vertical, left-diagonal, and right-diagonal line of symmetry (shown dashed in each example).

While translations also preserve the shape of the square, the labeled vertices will not change, so we consider any translation equivalent to the identity element e. Thus, there are eight symmetries of the square, including the identity transformation. We claim that these eight symmetry-preserving operations form a group, with multiplication $*$ defined to be the composition of two transformations. This group is known as the **dihedral group of degree 4**, and is denoted by D_4. To recap, the eight elements of D_4 are the symmetry transformations

$$\boxed{D_4 \; = \; \{e, \, R_{90}, \, R_{180}, \, R_{270}, \, H, \, V, \, D_{13}, \, D_{24}\}}.$$

To construct the multiplication table for D_4, we define the product $a * b$ to mean perform the transformation a on the original square first, and then perform transformation b on the result. For example, $R_{90} * R_{180}$ means rotate the original square clockwise 90° and then rotate again clockwise by 180°. The result is R_{270}. Similarly, $R_{180} * R_{270} = R_{90}$, because $180 + 270 = 450$ and R_{450} is equivalent to R_{90}.

Composing rotations and reflections is a bit more complicated. Using the labels on the vertices shown in Figures 1 and 2 helps determine the resulting product. It might also be illustrative to build a model and physically rotate and reflect the square. The figure below demonstrates the composition $H * R_{90} = D_{13}$. Note that the order of composition matters — D_4 is a non-commutative group! For instance, $H * R_{90} = D_{13}$ but $R_{90} * H = D_{24}$.

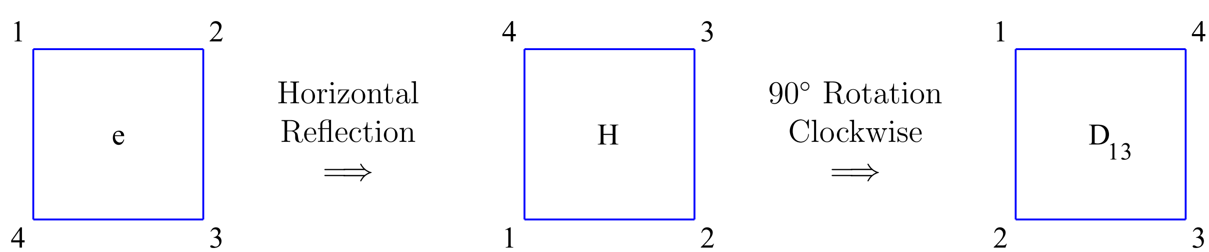

Figure 3: The product $H * R_{90} = D_{13}$.

Complete the multiplication table for D_4 below. A few rows have been provided to help you get started. Be sure to multiply the elements in the correct order: choose an element from the leftmost column first, and then compose it with an element from the top row. After you complete the table, confirm that D_4 is indeed a group. Give the inverse of each element.

$*$	e	R_{90}	R_{180}	R_{270}	H	V	D_{13}	D_{24}
e	e	R_{90}	R_{180}	R_{270}	H	V	D_{13}	D_{24}
R_{90}								
R_{180}								
R_{270}								
H	H	D_{13}	V	D_{24}	e	R_{180}	R_{90}	R_{270}
V								
D_{13}								
D_{24}								

Table 2: The multiplication table for the 8 symmetries of the square, D_4.

Example: The Musical Subgroup of D_4

Recall that the musical symmetries of inversion, retrograde, and retrograde-inversion correspond to the mathematical transformations of a horizontal reflection, a vertical reflection and a 180° rotation, respectively. Transpositions correspond to translations, which we identify with the identity element e as we did with the symmetries of the square. Consider the set of these four musical symmetries, $M = \{e, H, V, R_{180}\}$. There is something special about M.

Using your results from the group multiplication table for D_4, complete the multiplication table for M given below. Confirm that M, the set of musical symmetries, is a group on its own accord.

$*$	e	H	V	R_{180}
e				
H				
V				
R_{180}				

Table 3: The multiplication table for M, the four musical symmetries.

The special group M, which arises frequently in mathematics, is sometimes called the **Klein Four-Group**, named after the famous German mathematician Felix Klein (1849–1925). (There is even a "famous" mathematical a cappella group called "The Klein Four.") Since M is a subset of D_4, we call M a **subgroup** of D_4. In general, a subgroup of a group is a nonempty subset that forms a group by itself, under the *same* group operation. Subgroups play an important role in group theory and help us understand the structure of the larger group.