Math and Music Sampler

Gareth E. Roberts

Department of Mathematics and Computer Science
College of the Holy Cross
Worcester, MA

Topics in Mathematics: Math and Music

MATH 110 Spring 2018

January 23, 2018

Rhythm: Counting

Music is the pleasure the human soul experiences from counting without being aware that it is counting.
Gottfried Leibniz

Symbol:	𝅝	𝅗𝅥	♩	♪	𝅘𝅥𝅯	𝅘𝅥𝅰
Note:	whole	half	quarter	eighth	sixteenth	thirty-second
# of beats:	4	2	1	1/2	1/4	1/8

Table: The different types of notes and their durations, assuming that a quarter note equals one beat, form a geometric sequence.

Do a counting exercise here.

A polyrhythm is two distinct rhythmic patterns played simultaneously. Typically, each pattern is equally spaced.

These are common in many different types of music: Indian classical, jazz, African tribal music, modern classical (e.g. Stravinsky's *Rite of Spring*), even rock music!

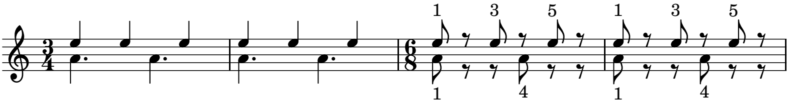

Figure: The three-against-two polyrhythm, where the top voice plays three equally spaced notes per measure while the bottom plays two. The last two measures show the same polyrhythm in $\frac{6}{8}$ time, demonstrating the precise location of each note.

$$\mathrm{lcm}(2, 3) = 6$$

Polyrhythm: Least Common Multiple

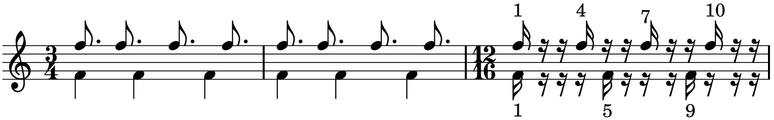

Figure: The four-against-three polyrhythm, where the top voice plays four equally spaced notes per measure while the bottom plays three. The last measure shows the same polyrhythm in $\frac{12}{16}$ time, demonstrating the precise location of each note.

$$\mathrm{lcm}(3,4) = 12$$

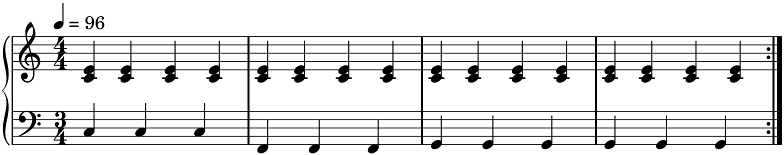

Figure: The primary piano part of The National's polyrhythmic hit *Fake Empire* (2008). The right hand plays in four while the left hand remains in three for the entire piece.

- Why do some combinations of pitches sound better than others?
- Why does the same note sound different on different instruments?
- How do we tune our instruments? Why are there 12 notes in the octave?

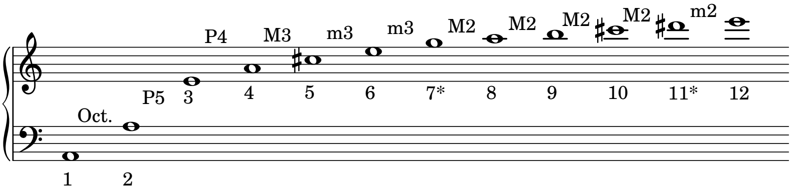

Figure: The overtone series for a low A.

Symmetry in Music: Group Theory

How to make a short motif go a long way:

Translations (shifting graph vertically) $\Longleftrightarrow$ Transpositions (shifting notes up or down)
Ex: Ballpark Music

Vertical Reflection (symmetry between right and left) $\Longleftrightarrow$ Retrograde (music same forward and backward)
Ex: Lean on Me

Horizontal Reflection (symmetry between top and bottom) $\Longleftrightarrow$ Inversion (what goes up, must come down)
Ex: Bach, Bach and more Bach

Symmetry in Music: Retrograde

Figure: Joseph Haydn, *Piano Sonata in A major* (Hob. XVI/26 or Landon 41, 1773), “Minuet in Reverse”

Change Ringing: An Example

1 2 3 4	1 3 4 2	1 4 2 3
2 1 4 3	3 1 2 4	4 1 3 2
2 4 1 3	3 2 1 4	4 3 1 2
2 4 3 1	3 2 4 1	4 3 2 1
4 2 3 1	2 3 4 1	3 4 2 1
4 2 1 3	2 3 1 4	3 4 1 2
4 1 2 3	2 1 3 4	3 1 4 2
1 4 3 2	1 2 4 3	1 3 2 4
		1 2 3 4

Canterbury Minimus (true extent on 4 bells)

There are $4! = 24$ different possible rows. Each must be rung exactly once starting and ending with rounds (1 2 3 4).

Both musical symmetry and change ringing involve the mathematical subject of group theory.