Math and Music: Understanding Pitch

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Pitch Perception

- Pitch is perceived. It is a sensation like hot or cold.
- Recall that frequency is measured in Hz (cycles per second). When we hear a note produced by an instrument, we are actually processing many different frequencies at once, called partials.
- The pitch perceived by our ear is called the fundamental frequency.
- Suppose we hear a note with fundamental frequency 200 Hz. This does not necessarily mean we are receiving a sine wave that oscillates 200 cycles per second. The sound wave may not be perfectly periodic, e.g., a bell or drum, or the fundamental could be entirely absent from the sound wave.

Residue Pitch

- Recall that frequencies in a 2 : 1 ratio are an octave apart.
- Experiment: What happens if we play two pure tones at 200 Hz and 400 Hz together?
- The perceived pitch depends on the relative amplitudes. If the amplitudes are the same, we hear the lower note as the primary pitch, 200 Hz. If we substantially decrease the amplitude of the 200 Hz tone, then the 400 Hz becomes the primary pitch.
- Experiment: What happens if we play 200, 400, 600, 800, and 1000 Hz tones simultaneously?
- Pitch perceived is still 200 Hz, but now the presence of the higher partials (integer multiples of the fundamental 200) begin to alter the character of the sound (its timbre).

Residue Pitch (cont.)

- Experiment: What happens if we play 400, 600, 800, and 1000 Hz tones simultaneously?
- Remarkably, the pitch perceived is 200 Hz, even though this frequency is not even present! We call 200 Hz the residue pitch, because it is what remains when the higher partials are combined together (Eric Heller, *Why You Hear What You Hear*, 2013).
- Experiment: What happens if we play 250, 300, 350, 400, 450, 500, and 550 Hz tones together?
- The perceived pitch is 50 Hz, two octaves below the note at 200 Hz!
- In each example, the residue pitch appears to be the greatest common divisor (GCD) of the set of frequencies. This is a good candidate for the frequency of the pitch perceived, but there are some exceptions (see text pp. 102–103).

Combining Sine Waves

What happens mathematically when we add two sine waves together?

Recall that for $y = a\sin(bt)$, the amplitude is a, the period is $2\pi/b$, and the frequency is $b/(2\pi)$.

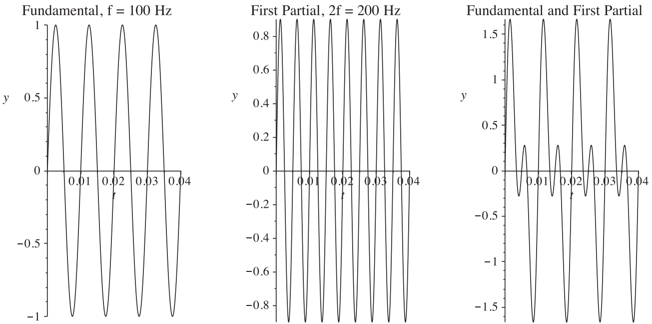

Figure: The graphs of the fundamental sine wave $y = \sin(2\pi \cdot 100t)$ (*left*), the first partial $y = 0.9\sin(2\pi \cdot 200t)$ (*center*), and the sum of the two waves (*right*). Notice that the period of the sum of the two waves, 0.01, is the same as that of the fundamental.

Combining the fundamental $f = 100$ with its first two partials $2f = 200$ and $3f = 300$.

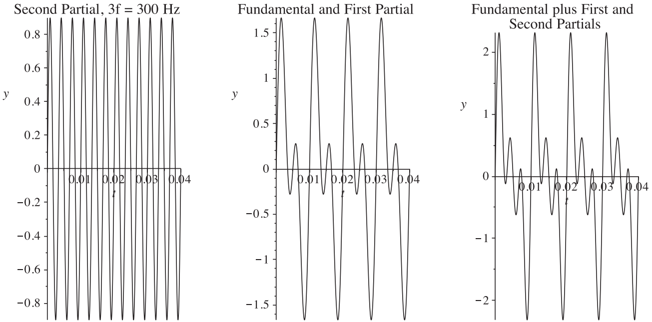

Figure: The graphs of the second partial $y = 0.9\sin(2\pi \cdot 300t)$ (*left*), the sum of the fundamental and first partial (*center*), and the sum of the fundamental and its first two partials (*right*). Notice that the period of the sum of the three waves, 0.01, is the same as that of the fundamental.

Removing the fundamental and first two partials does not effect the resulting period.

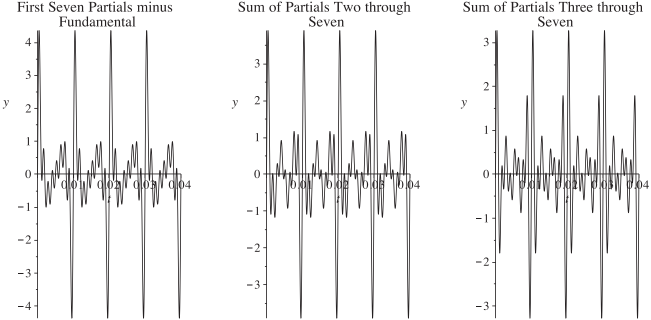

Figure: The graphs of the first seven partials of a 100 Hz wave with the fundamental frequency removed (*left*), the fundamental and the first partial removed (*center*), and the fundamental and the first two partials removed (*right*). In each case, the remaining frequencies produce a residue pitch at 100 Hz, as can be seen by the period of 0.01 in each graph.

The Overtone Series

- Why do different instruments playing the same pitch sound different to our ears?
- What causes the different character (timbre) of a sound? Why does an oboe sound different from a clarinet, even if they both are producing A440?
- Why do some combinations of pitches sound more consonant than others? Why does an octave sound stable and pleasing, while a tritone is unsettling?
- The answers involve the Overtone Series.

The Overtone Series (cont.)

The overtone series of a frequency f is simply the sequence of its consecutive integer multiples: $f, 2f, 3f, 4f, 5f, 6f, 7f, \ldots$.

Example: The overtone series for $f = 100$ Hz is

$$100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, \ldots .$$

The fundamental frequency is 100 Hz and the overtones (also called partials or harmonics) are 200 Hz, 300 Hz, 400 Hz, etc.

When a note is played on an instrument, the higher-frequency partials are also produced. These do not effect the fundamental frequency of the resulting sound wave, but they do effect its timbre.

Varying the strengths (the amplitudes) of the overtones creates different sounds.

The Highly Consonant Octave Interval

Consider the overtone series of both $f = 100$ Hz and $2f = 200$ Hz:

$$
\begin{array}{lcl}
f & : & 100,\ 200,\ 300,\ 400,\ 500,\ 600,\ 700,\ 800,\ 900,\ 1000,\ldots \\
2f & : & 200, 400, 600, 800, 1000, 1200, 1400, 1600, 1800, 2000, \ldots.
\end{array}
$$

The overtones in common are highlighted in red.

Key Observation: All of the partials for $2f$ are also included in the overtone series for f.

Since f and $2f$ have so many partials in common, playing them together leads to a very pleasing sound. This is why two notes an octave apart sound the "same;" the agreement of so many overtones helps create a resonance that is very satisfying to our brain.

Note: This argument works regardless of the particular value of f; it is the $2:1$ ratio between notes that is important.

The 3 : 2 Ratio

Consider the overtone series of both $f = 100$ Hz and $\frac{3}{2}f = 150$ Hz:

$$
\begin{array}{lcl}
f & : & 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, 1100, 1200, \ldots \\
\frac{3}{2}f & : & 150, 300, 450, 600, 750, 900, 1050, 1200, 1350, 1500, \ldots .
\end{array}
$$

Key Observation: Every other overtone of $\frac{3}{2}f$ agrees with every third partial of the original frequency f.

This consonance is also quite pleasing to our ear. It results in the musical interval of a perfect fifth.

General Rule: Combinations of pitches whose overtone series overlap tend to sound more consonant than those that do not (Hermann von Helmholtz (1821–1894)).

Play example of overtone series on piano.

Resonance

- When two objects vibrate at the same frequency, they are said to be in resonance.
- Resonance is very important in the study of acoustics. It serves to amplify sound so that it is more audible.
- Play tuning forks with amplifiers.
- Resonance can also be very dangerous. For example, if the frequency of walkers on a bridge match the natural frequency of the structure, it can oscillate wildly, even collapse! This is precisely what transpired on London's famous Millennium Bridge in 2000.

  Click Here for Video

Consider a simple oscillating system such as a mass attached to a spring. Normally, without any forcing or friction present, the solution is a simple periodic sine wave. However, if the system is forced periodically with frequency equal to the natural frequency of the oscillator, then resonance occurs and the amplitude grows linearly.

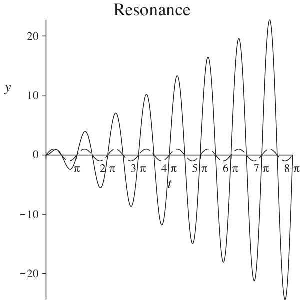

Figure: The graph of $y = \sin(2t)$ (*dashed*) versus $y = t\sin(2t)$ (*solid*), demonstrating the phenomenon of resonance.

Beats

What happens when two very close frequencies (not exact, but very close) are sounded simultaneously?

Example: A440 and A441

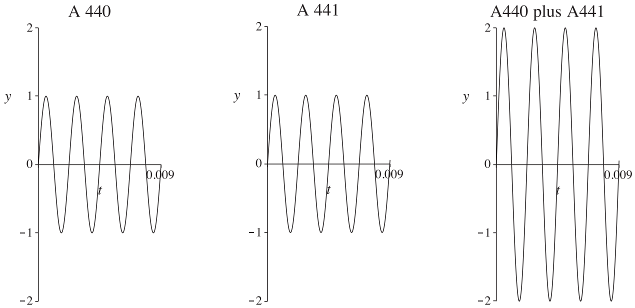

Figure: The graphs of A440 ($y = \sin(880\pi\, t)$), A441 ($y = \sin(882\pi\, t)$), and their sum over the first four cycles. Notice that the graph of the sum appears to be A440 with twice the amplitude.

Beats (cont.)

Example: A440 and A441

At first, the sum of the waves appears to be A440 with twice the amplitude. But what happens over a longer time interval?

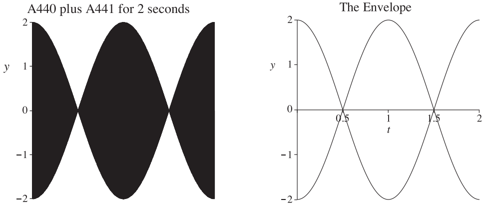

Figure: The graph of $y = \sin(880\pi\, t) + \sin(882\pi\, t) = 2\cos(\pi\, t)\sin(881\pi\, t)$ (*left*) over 2 seconds. The graph appears to be solid because the frequency of the wave is 440.5 Hz, implying that it has 881 cycles in 2 seconds. Notice that in 1 second, the resulting wave changes in amplitude from a maximum of 2 to a minimum of 0 (at $t = 0.5$) and then back to a maximum of 2. The graph of the wave's envelope, $y = \pm 2\cos(\pi\, t)$, is shown to the right, indicating one beat per second.

Theorem (The Rule of Beats)

Two pitches with close frequencies f and g Hz, with $f > g$, combine together to produce a pitch with the average frequency $(f+g)/2$ Hz, but they will beat at a frequency of $f - g$ Hz.

Example: Two waves with frequencies 300 Hz and 306 Hz will produce a pitch with frequency 303 Hz, with 6 beats per second.

Theorem follows from the trig identity (see pp. 110–111 in text for derivation)

$$\boxed{\sin u + \sin v \;=\; 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right).}$$

Example (cont.): Letting $u = 600\pi\, t$ and $v = 612\pi\, t$, we obtain

$$\sin(600\pi\, t) + \sin(612\pi t) \;=\; 2\cos(6\pi\, t)\sin(606\pi\, t).$$

This is a wave with frequency 303 Hz and 6 beats per second.