Math and Music: Polyrhythmic Music

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Polyrhythmic Music

- A polyrhythm is two distinct rhythmic patterns played simultaneously. Typically, each pattern is equally spaced.
- African tribal music is often polyrhythmic with different drums and percussion instruments playing different rhythms simultaneously.
- Classical Indian Music: tabla (pair of small hand drums). Drummers often play challenging combinations such as 11 beats in one hand and 12 in the other. Rhythmic patterns are a form of language mimicking syllables such as Dha, Tin, Na, Tun, or Ge.

Polyrhythm: 3 versus 2

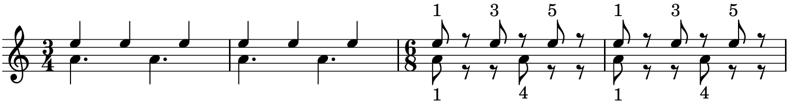


Figure: The three-against-two polyrhythm, where the top voice plays three equally spaced notes per measure while the bottom plays two. The last two measures show the same polyrhythm in $\frac{6}{8}$ time, demonstrating the precise location of each note. Phrase: "hot cup of tea"

Key mathematical idea is the least common multiple, denoted by lcm

$$\text{lcm}(2,3) \;=\; 6.$$

The least common multiple indicates how many parts to subdivide the measure into in order to determine the precise location of each note.

Polyrhythm: 4 versus 3

Figure: The four-against-three polyrhythm, where the top voice plays four equally spaced notes per measure while the bottom plays three. The last measure shows the same polyrhythm in $\frac{12}{16}$ time, demonstrating the precise location of each note.

$$\text{lcm}(3,4) = 12$$

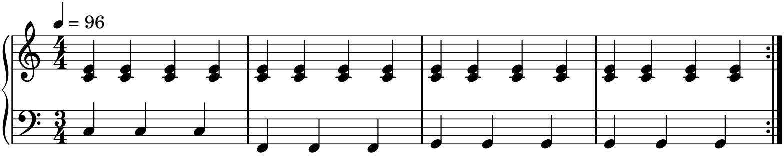


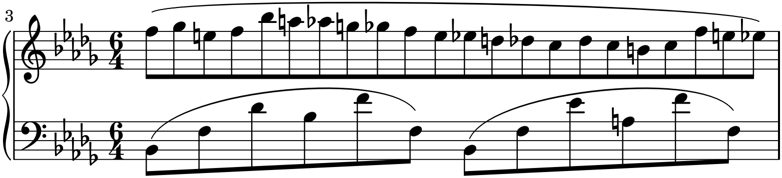
Figure: The primary piano part of The National's polyrhythmic hit *Fake Empire* (2008). The right hand plays in four while the left hand remains in three for the entire piece.

Polyrhythm: Chopin's Piano Music

Figure: Some difficult polyrhythms in measures 9 and 10 of Chopin's *Nocturne in B Major*, op. 9, no. 3 (1830–31). Bar 9 features two five-against-three polyrhythms, while the second half of measure 10 shows a seven-against-three pattern.

$$\text{lcm}(3, 5) = 15 \quad \text{and} \quad \text{lcm}(3, 7) = 21$$

Polyrhythm: Chopin



Exercise: Measure 3 of Chopin's *Nocturne in B-flat Minor*, op. 9, no. 1 (1830–31) features a particularly challenging polyrhythm. What is the smallest number of subdivisions needed to determine the precise location of each note?

$$\mathrm{lcm}(12, 22) = 132$$

Polyrhythm: Other Examples

- *The Rite of Spring*, Stravinsky (1913) has complex rhythmic patterns (e.g., unexpected accents, layering of different/competing ideas). Polyrhythms (three-against-two, four-against-three, and six-against-four) can be found in the movement *Cortége du Sage* (Procession of the Oldest and Wisest One).

- *Poème Symphonique*, György Ligeti (1962) is a piece for 100 metronomes, all set to randomly different tempos!
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- Rock and roll: *First Tube*, Phish; *Let Down*, Radiohead; *The Black Page*, Frank Zappa

The Least Common Multiple

How do we compute the least common multiple?

Method I: Write out the multiples of each number and then choose the smallest common multiple between them.

Example: $\text{lcm}(6, 8)$

Multiples of 6: $6, 12, 18, 24, 30, 36, 42, 48, \ldots$

Multiples of 8: $8, 16, 24, 32, 40, 48, 56, \ldots$

Common multiples: $24, 48, \ldots$ so the $\text{lcm}(6, 8) = 24$.

Note: Here, the answer is not the product of the two numbers.

Formula for the Least Common Multiple

Three Examples:

$$\text{lcm}(6,7) = 42, \qquad \text{lcm}(6,8) = 24, \qquad \text{lcm}(6,9) = 18.$$

What is different about the first case from the second two?
Answer: 6 and 7 have no common factors.

Definition

The greatest common factor of two natural numbers a and b is called the greatest common divisor, denoted as $\mathrm{GCD}(a,b)$. If two numbers a and b have no common factor other than 1, then $\mathrm{GCD}(a,b) = 1$. In this case, a and b are called relatively prime numbers.

Example: $(6,7)$, $(5,8)$ and $(11,60)$ are each pairs of relatively prime numbers.

Formula for the Least Common Multiple

Recall Three Examples:

$$\text{lcm}(6,7) = 42, \qquad \text{lcm}(6,8) = 24, \qquad \text{lcm}(6,9) = 18.$$

Greatest common divisors are:

$$\text{GCD}(6,7) = 1, \qquad \text{GCD}(6,8) = 2, \qquad \text{GCD}(6,9) = 3.$$

What's the connection? Look at the product $\text{GCD}(a,b) \cdot \text{lcm}(a,b)$.

$$\text{lcm}(a,b) \cdot \text{GCD}(a,b) = ab \implies \text{lcm}(a,b) = \frac{ab}{\text{GCD}(ab)}$$

Formula for the Least Common Multiple

$$\boxed{\text{lcm}(a, b) \;=\; \frac{ab}{\mathrm{GCD}(ab)}}$$

1. $\text{lcm}(9, 15) = 45$ because $\dfrac{9 \cdot 15}{3} \;=\; 3 \cdot 15 \;=\; 45.$

2. $\text{lcm}(12, 22) = 132$ because $\dfrac{12 \cdot 22}{2} \;=\; 6 \cdot 22 \;=\; 132.$

3. $\text{lcm}(24, 40) = 120$ because $\dfrac{24 \cdot 40}{8} \;=\; 3 \cdot 40 \;=\; 120.$