The Pythagorean Scale and Just Intonation

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Music is true. An octave is a mathematical reality. So is a 5th. So is a major 7th chord. And I have the feeling that these have emotional meanings to us, not only because we're taught that a major 7th is warm and fuzzy and a diminished is sort of threatening and dark, but also because they actually do have these meanings. It's almost like it's a language that's not a matter of our choosing. It's a truth. The laws of physics apply to music, and music follows that. So it really lifts us out of this subjective, opinionated human position and drops us into the cosmic picture just like that.

— *James Taylor*

Figure: Woodcut from Franchino Gafurio's *Theorica musice* (1492) indicating Pythagorean experiments with harmony and small integer ratios. Source: *Music and Mathematics: From Pythagoras to Fractals*, ed. Fauvel, Flood, Wilson, Oxford University Press (2003).

Figure: Jamming out on the wrenchophone at the Peabody Essex Museum. The ratios of the weights of the wrenches are small integer ratios like 2 : 1, 3 : 2, and 4 : 3.

Ratios Are Key

Key Fact: The musical interval between two notes is determined by the ratio between the frequencies of each note, not their difference.

Example: Recall that two notes whose frequencies are in a $2:1$ ratio are an octave apart. The musical interval between the notes corresponding to 100 and 200 Hz is an octave, as is the interval between 1500 and 3000 Hz. Even though the first pair is only a 100 Hz apart, and the second is 1500 Hz apart, both intervals are an octave.

Monochord Lab: The ratio between string lengths for a particular interval is the reciprocal of the one used between two frequencies. For instance, we cut the string length in half (multiply by $1/2$) to raise the pitch an octave, instead of doubling the frequency (multiplying by 2). Similarly, we multiply the length by $2/3$ to raise the pitch a P5, while we multiply the frequency by $3/2$.

The Pythagorean Scale

Scale Degree	Interval	Ratio
1	Uni.	1
2	M2	$9/8$
3	M3	$81/64$
4	P4	$4/3$
5	P5	$3/2$
6	M6	$27/16$
7	M7	$243/128$
8 = 1	Oct.	$2/1$

Table: The ratios or multipliers used to raise a note (increase the frequency) by a given musical interval in the Pythagorean scale. These values are the reciprocals of those found for the Monochord Lab.

The Pythagorean Scale (cont.)

Example: To find the frequency of the note that is a perfect fourth above 300 Hz, we multiply 300 by $4/3$, giving a frequency of 400 Hz,

$$300 \cdot \frac{4}{3} = 100 \cdot 4 = 400\,.$$

Exercise: Find the frequency of the note a major sixth above 320 Hz.
Answer: ________

Exercise: According to the Pythagorean scale, what is the musical interval between 400 and 450 Hz?
Answer: ________

The Pythagorean Scale: Key Facts

1. The entire scale is generated from two ratios: the $2:1$ octave and the $3:2$ perfect fifth. Consequently, the numerator and denominator of each ratio are always a power of 2 or 3.
2. All perfect fifths in the scale are in the ratio $3:2$.
3. All five whole steps in the Pythagorean major scale are in the ratio $W = 9:8$.
4. Both half steps in the Pythagorean major scale are in the ratio $H = 256:243$.
5. Two half steps do not equal one whole step, that is, $H^2 \neq W$. This is a big problem!

The Pythagorean Comma

Generally speaking, a comma is a gap or discrepancy.

$$\frac{W}{H^2} = \frac{9/8}{(256/243)^2} = \frac{9 \cdot 243 \cdot 243}{8 \cdot 256 \cdot 256} = \frac{3^{12}}{2^{19}} = \frac{531{,}441}{524{,}288} \approx 1.01364.$$

The fact that $W \neq H^2$ presents huge problems when playing notes outside the major scale. For instance, going from C to C$\sharp$ and then C$\sharp$ to D (two half steps) is different than going from C to D (one whole step).

Definition

The Pythagorean comma is defined to be the ratio $3^{12} : 2^{19}$. It equals the gap between two half steps and one whole step in the Pythagorean scale.

The Circle of Fifths

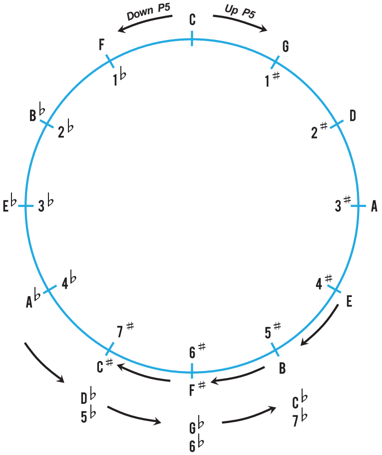


Figure: The circle of fifths. Going up by 12 perfect fifths is equivalent to going up by 7 octaves. Mathematically speaking, $12 \cdot 7 = 7 \cdot 12 = 84$.

The Circle of Fifths via Pythagorean Tuning

Recall: an octave is a 2 : 1 ratio and a P5 is a 3 : 2 ratio.
Thus, it should be the case that

$$\left(\frac{3}{2}\right)^{12} \stackrel{?}{=} 2^7, \quad \text{or} \quad 3^{12} \stackrel{?}{=} 2^{19}.$$

But this is clearly false! A power of 3 is always an odd number, while a power of 2 is even.

The circle of fifths does not close up using Pythagorean tuning; it is more like a spiral of fifths. The gap when going around the circle by 12 perfect fifths is precisely a Pythagorean comma, $3^{12}/2^{19}$, above the correct note (7 octaves).

The Spiral of Fifths

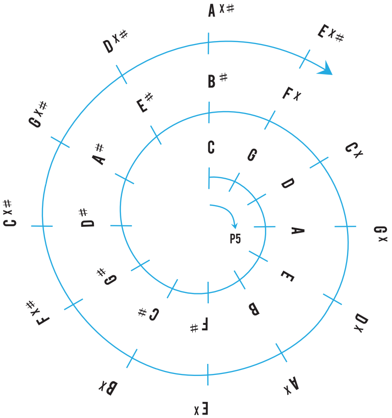


Figure: For the Pythagorean scale, the circle of fifths does not close up; it spirals around repeatedly, creating a new gap equal to the Pythagorean comma each time it passes by C.

The Pythagorean Scale: Drawbacks

1. Two half steps do not equal one whole step (Pythagorean comma).
2. Hard to consistently tune notes outside major scale. How do we play chromatic melodies? How do we change keys?
3. Circle of fifths does not close up. This leads to peculiarities where enharmonic equivalents are no longer equivalent, e.g., $B\sharp \neq C$ or $F\sharp \neq G\flat$.

The Overtone Series Revisited

Recall: The overtone series of a frequency f is simply the sequence of its consecutive integer multiples: $f, 2f, 3f, 4f, 5f, 6f, 7f, \ldots$.

Example: The overtone series for $f = 110$ Hz is

$$110, 220, 330, 440, 550, 660, 770, 880, 990, 1100, 1210, 1320, \ldots.$$

The fundamental frequency is 110 Hz and the overtones (also called partials or harmonics) are 220 Hz, 330 Hz, 440 Hz, etc.

When a note is played on an instrument, the higher-frequency partials are also produced. These do not effect the fundamental frequency of the resulting sound wave, but they do effect its timbre.

Varying the strengths (the amplitudes) of the overtones creates different sounds.

Consonant Intervals

General Rule: Combinations of pitches whose overtone series overlap tend to sound more consonant than those that do not (Hermann von Helmholtz (1821–1894)).

Example: The octave.

$$\begin{array}{lcl} f & : & 100,\ 200,\ 300,\ 400,\ 500,\ 600,\ 700,\ 800,\ 900,\ 1000, \ldots \\ 2f & : & 200, 400, 600, 800, 1000, 1200, 1400, 1600, 1800, 2000, \ldots. \end{array}$$

The overtones in common are highlighted in red.

Example: The perfect fifth.

$$\begin{array}{lcl} f & : & 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, 1100, 1200, \ldots \\ \frac{3}{2}f & : & 150, 300, 450, 600, 750, 900, 1050, 1200, 1350, 1500, \ldots. \end{array}$$

The Overtone Series

Using the ratios of the Pythagorean scale, we can determine the corresponding notes of the overtone series for a fixed fundamental. Some of these fit exactly into the scale or are close enough, but others (e.g., $7f$ and $11f$) are substantially far from the Western (chromatic) scale.

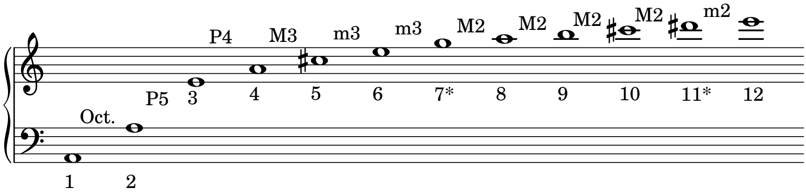


Figure: The first 12 notes corresponding to the overtone series of $f =$ A110 Hz, along with the intervals between consecutive notes. The notes indicated for $7f$ and $11f$ are approximations. Both of these overtones are flatter than the note indicated on the staff.

Also sprach Zarathustra

The Intervals of the Overtone Series

Note: The intervals between successive notes in the overtone series are always the same because they are determined by ratios. Thus, the frequency of the fundamental is irrelevant; the ensuing overtones always follow the exact same interval pattern:

$$\text{Oct., P5, P4, M3, m3, m3}^*\text{, M2, M2, M2, M2}^*\text{, m2,}$$

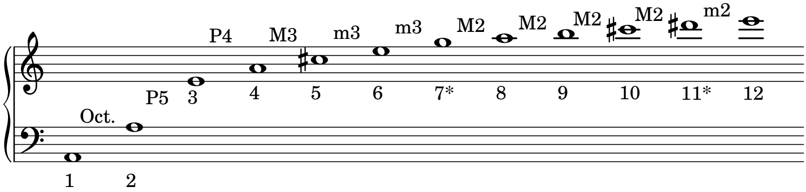
where the $*$ indicates the fact that the 7th and 11th overtones are approximations.

The successive ratios between the first 12 notes are

$$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}, \frac{10}{9}, \frac{11}{10}, \frac{12}{11}.$$

Those in red agree precisely with the Pythagorean scale.

The Major Chord



Observation 1: The first six notes of the overtone series, as well as notes 8, 10, and 12, are the pitches of a major triad (here, A, C♯, and E). A appears four times, E appears three times, and C♯ twice. This helps explain why a major chord sounds so pleasing to our ear.

Observation 2: The ratio for a major triad is simply $4:5:6$, since these are the ratios that correspond to $4f, 5f$, and $6f$ (A440, C♯550, and E660 in our example). In other words, a major third is $5:4$ and a minor third is $6:5$. Why not create a scale that emphasizes these ratios instead of those from the Pythagorean scale?

Just Intonation

Scale Degree	Interval	Ratio
1	Uni.	1
2	M2	$9/8$
3	M3	$5/4$
4	P4	$4/3$
5	P5	$3/2$
6	M6	$5/3$
7	M7	$15/8$
8 = 1	Oct.	$2/1$

Table: The ratios or multipliers used to raise a note (increase the frequency) by a given musical interval in just intonation. The ratios in red indicate changes from the Pythagorean scale. The lower-numbered ratios are in excellent agreement with the ratios of the overtone series.

One of the strengths of just intonation is that the major chords I, IV, and V are all in the simple $4:5:6$ ratio, leading to a particularly pleasing consonance because of its alignment with the overtone series.

Example: To find the note that is a major third above 200 Hz using just intonation (called a just major third), we multiply 200 by $5/4$, giving a frequency of 250 Hz.

$$200 \cdot \frac{5}{4} = 50 \cdot 5 = 250 .$$

Exercise: Find the frequency of the note a just major sixth above 300 Hz. Answer: ________

Exercise: According to just intonation, what is the musical interval between 800 and 1000 Hz?
Answer: _____________

The Syntonic Comma

What is a whole step in just intonation?

- $W = 9/8$ since that is the ratio between scale degrees one and two.

- On the other hand, $W = 10/9$ since that is the ratio between scale degrees two and three ($5/4 \div 9/8 = 10/9$). Two different whole steps?!

Definition

The syntonic comma is defined to be the ratio $81:80$. It measures the gap (as a ratio) between the two different whole steps in the just intonation tuning system. It is also the gap between the three contrasting ratios (major third, sixth, and seventh) between the Pythagorean scale and just intonation.

Just Intonation: Halfstep

What is a half step in just intonation?

- Between scale degrees three and four, we have $\frac{5}{4} \cdot H = \frac{4}{3}$, which means $H = \frac{4}{3} \cdot \frac{4}{5} = \frac{16}{15}$.
- Between scale degrees seven and eight, we have $\frac{15}{8} \cdot H = \frac{2}{1}$, which means $H = \frac{2}{1} \cdot \frac{8}{15} = \frac{16}{15}$.

However, two half steps do not equal either whole step:

$$H^2 = \frac{256}{225} = 1.13\bar{7} > \frac{9}{8} = 1.125 > \frac{10}{9} = 1.11\bar{1}.$$

This means there are three different ways to raise the pitch a whole step in just intonation.

Just Intonation: Drawbacks

1. Two different whole steps (syntonic comma) and two half steps do not equal either whole step.
2. Hard to consistently tune notes outside major scale. How do we play chromatic melodies? How do we change keys?
3. Circle of fifths still does not close up because the ratios for the octave and perfect fifth are the same as the Pythagorean scale.
4. A simple melody can go flat or sharp when returning to the original note. This is called melodic drift.

Just Intonation: Melodic Drift

Figure: A simple melody that drifts downward under just intonation.

Let f be the frequency of the starting C. Using the ratios of just intonation, the frequency of the final C is

$$f \cdot \frac{4}{3} \cdot \frac{5}{6} \cdot \frac{4}{3} \cdot \frac{2}{3} \;=\; \frac{80}{81} f,$$

which is a full syntonic comma lower than the opening C.

Dunne and McConnell describe a similar drift in the chorus of Madonna's *Borderline*. Assuming the band and singer are using just intonation, the entire song drifts upward by more than a half step.

Major versus Minor Chord

What are some physical explanations for why a major chord sounds so triumphant and happy, while a minor chord is sadder and more somber?

1. Chord ratios.

Major chord ratio: $\frac{1}{1}, \frac{5}{4}, \frac{3}{2}$ or $\frac{4}{4}, \frac{5}{4}, \frac{6}{4} \Longrightarrow 4:5:6.$

Minor chord ratio: $\frac{1}{1}, \frac{6}{5}, \frac{3}{2}$ or $\frac{10}{10}, \frac{12}{10}, \frac{15}{10} \Longrightarrow 10:12:15.$

The higher-numbered ratio for the minor chord means less overlap in the corresponding overtone series, which in turn implies less consonance in the chord. (HW)

Major versus Minor Chord (cont.)

2. Residue Pitch.

Major chord: The residue pitch for frequencies $4f, 5f$, and $6f$ is just f, which is two octaves below the root of the chord. The overtones of the residue pitch nicely reinforce the notes of the major triad.

Minor chord: The residue pitch for frequencies $10f, 12f$, and $15f$ is again f, but this note is three octaves and a major third below the root of the chord. It is not contained in the original chord. Even if it is audible, it does not reinforce the notes in the minor triad.

For a minor chord, the pitch most likely to contribute to our perception is the first overtone common of all three pitches. This note is two octaves and a fifth above the root, e.g., in a C minor triad C, E$\flat$, G, it would be the note G". Because this matches the fifth, not the tonic, and because it is well above the notes in the chord, it offers no real support.