MATH 134 Calculus 2 with FUNdamentals

Section 10.2: Summing an Infinite Series

SOLUTIONS

Exercise 1: Consider the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1}$. Write out the first five terms in the series and then compute the first five partial sums: s_1, s_2, s_3, s_4, and s_5. Does the series converge or diverge? Explain.

Answer: This series diverges. Let's start by writing out the first six terms of the series. These can be found by plugging in $n = 1, 2, 3, 4, \ldots$ in order.

$$\sum_{n=1}^{\infty}(-1)^{n+1} \;=\; 1-1+1-1+1-1+-\cdots$$

The partial sum s_n is found by summing the terms from a_1 to a_n:

$$\begin{aligned}
s_1 &= a_1 \;=\; 1\\
s_2 &= a_1+a_2 \;=\; 1-1 \;=\; 0\\
s_3 &= a_1+a_2+a_3 \;=\; 1-1+1 \;=\; 1\\
s_4 &= a_1+a_2+a_3+a_4 \;=\; 1-1+1-1 \;=\; 0\\
s_5 &= a_1+a_2+a_3+a_4+a_5 \;=\; 1-1+1-1+1 \;=\; 1
\end{aligned}$$

We see that the sequence of partial sums is simply $1, 0, 1, 0, 1, 0, \ldots$. Since this sequence never settles down to a particular value, it diverges. By definition, since the sequence of partial sums diverges, so does the infinite series.

Exercise 2: Find the sum of each of the following geometric series or state that the series diverges.

(a) $2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\cdots$

(b) $8-2+\frac{1}{2}-\frac{1}{8}+\frac{1}{32}-+\cdots$

(c) $\sum_{n=1}^{\infty}\left(\frac{-3}{5}\right)^n$

(d) $\sum_{n=0}^{\infty}\left(\frac{\pi}{3}\right)^n$

Answer: **(a)** This series is geometric with a ratio of $r = 2/3$. This can be found by inspection (e.g., $2\cdot\frac{2}{3}=\frac{4}{3}$), or by dividing any term in the series by the previous term (e.g., $\frac{8}{9}\div\frac{4}{3}=\frac{8}{9}\cdot\frac{3}{4}=\frac{2}{3}$). Since

$r = 2/3 < 1$, the geometric series converges. The sum is

$$S = \frac{a}{1-r} = \frac{2}{1-\frac{2}{3}} = \frac{2}{\frac{1}{3}} = 6.$$

(b) This series is geometric with a ratio of $r = -1/4$. This can be found by inspection (e.g., $8 \cdot -\frac{1}{4} = -2$), or by dividing any term in the series by the previous term (e.g., $-\frac{1}{8} \div \frac{1}{2} = -\frac{1}{8} \cdot \frac{2}{1} = -\frac{1}{4}$). Since $|r| = |-1/4| = 1/4 < 1$, the geometric series converges. The sum is

$$S = \frac{a}{1-r} = \frac{8}{1-(-\frac{1}{4})} = \frac{8}{\frac{5}{4}} = \frac{32}{5}.$$

(c) If we write out the first few terms of this series, we find

$$\sum_{n=1}^{\infty} \left(\frac{-3}{5}\right)^n = -\frac{3}{5} + \frac{9}{25} - \frac{27}{125} + - \cdots$$

This series is geometric with a ratio of $r = -3/5$ (the number in the parentheses). This can be found by inspection (e.g., $-\frac{3}{5} \cdot -\frac{3}{5} = \frac{9}{25}$), or by dividing any term in the series by the previous term (e.g., $-\frac{27}{125} \div \frac{9}{25} = -\frac{27}{125} \cdot \frac{25}{9} = -\frac{3}{5}$). Since $|r| = |-3/5| = 3/5 < 1$, the geometric series converges. The sum is

$$S = \frac{a}{1-r} = \frac{-\frac{3}{5}}{1-(-\frac{3}{5})} = \frac{-\frac{3}{5}}{\frac{8}{5}} = -\frac{3}{8}.$$

(d) This series diverges. If we write out the first few terms of this series, we find

$$\sum_{n=0}^{\infty} \left(\frac{\pi}{3}\right)^n = 1 + \frac{\pi}{3} + \frac{\pi^2}{9} + \frac{\pi^3}{27} + \cdots$$

This series is geometric with a ratio of $r = \pi/3$ (the number in the parentheses). This can be found by inspection (e.g., $\frac{\pi}{3} \cdot \frac{\pi}{3} = \frac{\pi^2}{9}$), or by dividing any term in the series by the previous term (e.g., $\frac{\pi^3}{27} \div \frac{\pi^2}{9} = \frac{\pi^3}{27} \cdot \frac{9}{\pi^2} = \frac{\pi}{3}$). Since $|r| = \pi/3 > 1$, the geometric series diverges (the terms grow in size quickly).

Exercise 3: Use the nth term test to explain why the following series diverge:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3n-1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

(c) $\cos 1 + \cos\frac{1}{2} + \cos\frac{1}{3} + \cos\frac{1}{4} + \cdots$

Answer: **(a)** We compute

$$\lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}$$

using L'Hôpital's Rule. Since this limit is not zero, the series diverges by the nth term test. In essence, the terms in the series are approaching $1/3$, so we are repeatedly summing $1/3$ over and over again. The sequence of partial sums is approaching ∞ and so the series diverges.

(b) To compute the limit, we divide top and bottom of the fraction by the highest power n, except on the bottom of the fraction we divide by $\sqrt{n^2}$ (which is the same as n). We have

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} \;=\; \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{\sqrt{n^2+1}}{\sqrt{n^2}}} \;=\; \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} \;=\; \frac{1}{\sqrt{1+0}} \;=\; 1\,.$$

Since this limit is not zero, the series diverges by the nth term test. In essence, the terms in the series are approaching 1, so we are repeatedly summing 1 over and over again. The sequence of partial sums is approaching ∞ and so the series diverges.

(c) The terms are of the form $\cos\left(\frac{1}{n}\right)$. We compute

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) \;=\; \cos\left(\lim_{n\to\infty}\frac{1}{n}\right) \;=\; \cos(0) \;=\; 1$$

using the fact that the cosine function is continuous. Since this limit is not zero, the series diverges by the nth term test. In essence, the terms in the series are approaching 1, so we are repeatedly summing 1 over and over again. The sequence of partial sums is approaching ∞ and so the series diverges.