MATH 134 Calculus 2 with FUNdamentals

Section 10.1: Sequences

SOLUTIONS

Exercise 1: Let a_n be the sequence defined by the relation $a_{n+1} = a_n^2 - 1$ and $a_1 = 2$. What are the next four terms in the sequence?

Answer: This is a recursive sequence where the next term a_{n+1} is found by squaring the previous term a_n and subtracting 1.

We start with $a_1 = 2$. Then $a_2 = a_1^2 - 1 = 2^2 - 1 = 3$. Next, $a_3 = a_2^2 - 1 = 3^2 - 1 = 8$. Then $a_4 = a_3^2 - 1 = 8^2 - 1 = 63$. Finally, $a_5 = a_4^2 - 1 = 63^2 - 1 = 3968$.

Thus the sequence beginsq

$$2, 3, 8, 63, 3968, \ldots$$

The sequence is heading off to ∞ so it diverges.

Exercise 2: Consider the two sequences defined by $a_n = (\frac{2}{3})^n$ and $b_n = (\frac{3}{2})^n$. Write out the first four terms of each sequence and then determine whether each sequence converges or diverges. If it converges, state the limit of the sequence.

Answer: Each sequence is an example of a geometric sequence. The first four terms of a_n are $\dfrac{2}{3}, \dfrac{4}{9}, \dfrac{8}{27}, \dfrac{16}{81}$. Notice that the next term is found by multiplying the previous term by the common ratio $r = 2/3$. Since $r < 1$, the terms are getting smaller and smaller and approaching 0. In other words, the sequence converges to 0.

In contrast, the first four terms of b_n are $\dfrac{3}{2}, \dfrac{9}{4}, \dfrac{27}{8}, \dfrac{81}{16}$. Notice that the next term is found by multiplying the previous term by the common ratio $r = 3/2$. Since $r > 1$, this sequence is approaching ∞, which means the sequence diverges.

Exercise 3: Find a formula for each of the following sequences (e.g., $a_n = 2n^2 - 3$).

(a) 2, 4, 6, 8, 10, 12, ...

Answer: $a_n = 2n$. It is also acceptable to write $a_{n+1} = a_n + 2, a_1 = 2$.

(b) $1, \dfrac{1}{3}, \dfrac{1}{9}, \dfrac{1}{27}, \dfrac{1}{81}, \ldots$

Answer: $a_n = \dfrac{1}{3^{n-1}}$ (notice the powers of 3 in the denominators).

(c) $\dfrac{1}{2}, \dfrac{1}{5}, \dfrac{1}{10}, \dfrac{1}{17}, \dfrac{1}{26}, \ldots$

Answer: $a_n = \dfrac{1}{n^2 + 1}$ (the denominators are one bigger than the perfect squares).

Exercise 4: For each of the following sequences, write out the first four terms of the sequence. Then determine whether the sequence converges or diverges. If it converges, state the limit of the sequence.

(a) $a_n = \dfrac{1}{n}$

Answer: The first four terms are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. Since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, the sequence converges to 0.

(b) $b_n = \left(\dfrac{3}{\pi}\right)^n$

Answer: This is a geometric sequence with ratio $r = 3/\pi$. Note that $r < 1$. The first four terms are $\frac{3}{\pi}, \frac{9}{\pi^2}, \frac{27}{\pi^3}, \frac{81}{\pi^4}$. Since $\lim\limits_{n\to\infty} \left(\frac{3}{\pi}\right)^n = 0$, the sequence converges to 0.

(c) $c_n = \left(\dfrac{\pi}{3}\right)^n$

Answer: This is a geometric sequence with ratio $r = \pi/3$. Note that $r > 1$. The first four terms are $\frac{\pi}{3}, \frac{\pi^2}{9}, \frac{\pi^3}{27}, \frac{\pi^4}{81}$. Since $\lim\limits_{n\to\infty} \left(\frac{\pi}{3}\right)^n = \infty$, the sequence diverges.

Key Fact: A geometric sequence with ratio r satisfying $|r| < 1$ converges to 0. If $r = 1$, the sequence converges to whatever the starting value is. Otherwise the sequence diverges.

(d) $a_n = \dfrac{n}{\sqrt{4n^2+3}}$

Answer: The first four terms are $\frac{1}{\sqrt{7}}, \frac{2}{\sqrt{19}}, \frac{3}{\sqrt{39}}, \frac{4}{\sqrt{67}}$. As $n \to \infty$, the 3 in the denominator becomes irrelevant when compared to $4n^2$. Since

$$\frac{n}{\sqrt{4n^2}} = \frac{n}{2n} = \frac{1}{2},$$

we have $\lim\limits_{n\to\infty} \frac{n}{\sqrt{4n^2+3}} = \frac{1}{2}$. Thus, the sequence converges to $1/2$.

(e) $a_n = \dfrac{n}{2^n}$

Answer: The first four terms are $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}$. Using L'Hôpital's Rule, we have

$$\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{\ln 2 \cdot 2^n} = 0\,.$$

Thus, the sequence converges to 0.

(f) $s_n = \sin(n)$

Answer: This sequence hops around the unit circle, identifying the y-value at successive angles one radian apart (recall that one radian is approximately $57.3°$). The first four terms are $\sin(1), \sin(2), \sin(3), \sin(4)$. Since $\lim\limits_{n\to\infty} \sin(n)$ does not exist, the sequence diverges.