MATH 134 Calculus 2 with FUNdamentals

Section 5.3: The Indefinite Integral

This section focuses on using the differentiation process in the opposite direction: instead of finding the derivative of a function f, we find a function F whose derivative equals f. The function F is known as an antiderivative of f ("anti" = "opposite").

Definition: If $F'(x) = f(x)$ over an interval (a, b), then F is called an **antiderivative** of f.

Example 1: $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ because $\dfrac{d}{dx}\left(x^2\right) = 2x$. (Recall that $\frac{d}{dx}$ means to take the derivative with respect to x.)

Example 2: $G(x) = \sin x$ is an antiderivative of $g(x) = \cos x$ because $\dfrac{d}{dx}(\sin x) = \cos x$.

Some Important Notes:

- Finding an antiderivative means going backwards (like an inverse). What function, when you take its derivative, produces the given function? We typically use **capital letters** to represent antiderivatives.
- To check whether F is an antiderivative of f, simply take the derivative of F and see if it equals f. In other words, check that $F'(x) = f(x)$.
- Notice that $F(x) = x^2 + 5$ and $F(x) = x^2 - \pi$ are also antiderivatives of $f(x) = 2x$ because the derivative of a constant is zero. In general, if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + c$ for **any** constant c. There are an **infinite** number of antiderivatives for any one given function.

Exercise 1: Find **two** antiderivatives for $f(x) = \sin x$.

The Indefinite Integral

The process of finding an antiderivative is called **integration**. We use the integral sign $\int$, without any limits of integration, to indicate a general antiderivative. In other words,

$$\int f(x)\,dx \;=\; F(x) + c \quad \text{means that} \quad F'(x) = f(x).$$

The expression $\displaystyle\int f(x)\,dx$ is called an **indefinite integral**. Note that there are *no* limits of integration. The reason why we use the same symbol for an antiderivative as we did for the signed area under the curve (the definite integral) will become clear when we study the Fundamental Theorem of Calculus.

Exercise 2: Check that $\displaystyle\int 12x^5\,dx = 2x^6 + c$ and $\displaystyle\int \cos(3\theta)\,d\theta = \frac{1}{3}\sin(3\theta) + c$ by taking derivatives.

Key Integration Formulas: $c, k \in \mathbb{R}$ are arbitrary constants

1. $\displaystyle\int 0\,dx = c$, where c is an arbitrary constant

2. $\displaystyle\int k\,dx = kx + c$

3. **Power Rule**: $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad \text{where } n \neq -1$

4. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + c$ (notice the absolute value sign!)

5. $\displaystyle\int e^x\,dx = e^x + c$ and more generally, $\displaystyle\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c, \quad (k \neq 0)$

6. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + c$ for any real number $a > 0$

7. $\displaystyle\int \sin x\,dx = -\cos x + c$ and more generally, $\displaystyle\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + c, \quad (k \neq 0)$

8. $\displaystyle\int \cos x\,dx = \sin x + c$ and more generally, $\displaystyle\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + c, \quad (k \neq 0)$

9. $\displaystyle\int \sec^2 x\,dx = \tan x + c$

10. $\displaystyle\int \csc^2 x\,dx = -\cot x + c$

11. $\displaystyle\int \sec x \tan x\,dx = \sec x + c$

12. $\displaystyle\int \csc x \cot x\,dx = -\csc x + c$

13. **Linearity: (i)** $\displaystyle\int kf(x)\,dx \;=\; k\int f(x)\,dx$ (constants pull out)

    (ii) $\displaystyle\int f(x) + g(x)\,dx \;=\; \int f(x)\,dx + \int g(x)\,dx$ (integral of a sum is the sum of the integrals)

Exercise 3: Notice the absolute value signs in Formula 4. Why are they needed? Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0\,. \end{cases}$$

Using this definition and the chain rule, check that Formula 4 is correct. In other words, check that $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$ when $x > 0$ *and* when $x < 0$.

Exercise 4: Use the chain rule to verify Formula 5:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c \qquad (k \text{ is a constant})$$

Exercise 5: Find each indefinite integral using the correct formulas.

(a) $\displaystyle\int 6\sin(3x) + 12\cos(4x)\,dx$

(b) $\displaystyle\int \frac{4}{t^3} + 9e^{-3t}\,dt$

Exercise 6: Find an antiderivative $F(x)$ of $f(x) = \dfrac{1}{\sqrt{x}} + 14x^6$ satisfying $F(1) = 0$.

Note: For homework purposes, this is the same thing as solving the initial value problem

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} + 14x^6, \quad y(1) = 0$$

(see Exercises 47–62 on p. 287 of the textbook). The goal here is to find the specific c-value for the antiderivative that satisfies $F(1) = 0$.

Exercise 7: Suppose that $f''(x) = -4\sin x + 12x^3$ and $f'(0) = 10$ and $f(0) = \pi$. Find $f(x)$.

Hint: First find $f'(x)$, then find $f(x)$, computing the correct c-values along the way to ensure that $f'(0) = 10$ and $f(0) = \pi$.

Exercise 8: A particle moves on a line with acceleration given by $a(t) = 4t + 6$ m/s^2. If the initial velocity is $v(0) = -7$ m/s and the initial position is $s(0) = 25$ m, find the position function $s(t)$.

Hint: The acceleration $a(t)$ is defined as the derivative of velocity, so $a(t) = v'(t)$. Thus, integrating $a(t)$ once yields $v(t)$. Since $v(t) = s'(t)$ (velocity is the derivative of position), another integration returns us to $s(t)$. Simply put, $a(t) = s''(t)$. Be sure to find the correct c-values along the way.