MATH 134 Calculus with FUNdamentals 2

Section 6.1: Area Between Two Curves

In Chapter 6 we will learn about some geometric applications of the integral. This section is focused on finding the area between two curves.

Area Between Two Curves: Vertically Simple

Suppose we have two functions $f(x)$ and $g(x)$ each defined over an interval $[a, b]$ with $f(x) \geq g(x)$ on the entire interval. This means the graph of f lies above the graph of g (they could touch in a few places). To find the area between the two graphs, we simply subtract the area under f from the area under g (see Figure 3).

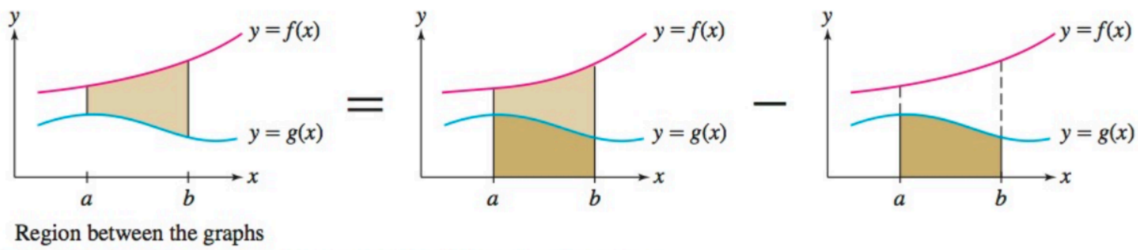

FIGURE 3 The area between the graphs is a difference of two areas.

Using linearity of the integral, the area between the graphs of f and g is

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \;=\; \int_a^b [f(x) - g(x)]\,dx\,.$$

Thus, assuming that the graph of f lies *above* the graph of g, we have

$$\boxed{\text{Area between the graphs of } f \text{ and } g \;=\; \int_a^b [f(x) - g(x)]\,dx\,.} \tag{1}$$

Note that this value should be positive because we have assumed that $f(x) \geq g(x)$ over the entire interval. Also, this formula works for regions that are **vertically simple**, where any vertical line through the region first intersects f (coming from above) and then leaves intersecting g (below).

Example 1: Find the area of the region enclosed by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant.

Answer: There are two things we need to determine in order to set up the integral: where do the curves intersect and which curve is above the other? To find the points of intersection we solve the equation $\sqrt{x} = x$. Squaring both sides we obtain $x = x^2$ or $x^2 - x = 0$. Factoring the left-hand side, we find $x(x-1) = 0$ so that $x = 0$ and $x = 1$ are the points of intersection. A graph of the two functions $f(x) = \sqrt{x}$ and $g(x) = x$ indicates that $y = \sqrt{x}$ lies above $y = x$ when $x \in [0, 1]$. This can also be checked by plugging in a few x-values, e.g., $\sqrt{1/4} = 1/2 > 1/4$ or $\sqrt{4/9} = 2/3 > 4/9$ (see Figure 1 on the next page).

Using formula (1) above, we compute the area of the region to be

$$\int_0^1 \sqrt{x} - x\,dx \;=\; \int_0^1 x^{1/2} - x\,dx \;=\; \frac{2}{3}x^{3/2} - \frac{1}{2}x^2\bigg|_0^1 \;=\; \frac{2}{3} - \frac{1}{2} - (0 - 0) \;=\; \frac{1}{6}\,.$$

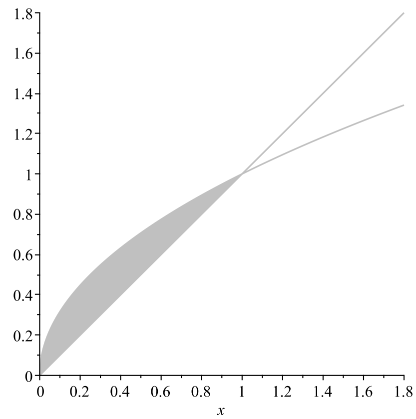

Figure 1: The region between $y = \sqrt{x}$ and $y = x$ for $0 \leq x \leq 1$ is shaded. Note that $y = \sqrt{x}$ lies above $y = x$ on this interval.

Exercise 1: Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 8 - x^2$. Sketch the region.

Exercise 2: Find the area of the region enclosed by the graphs of the functions $f(x) = x^3 - x$ and $g(x) = 8x$. **Hint:** You will need to set up two integrals. Why?

Area Between Two Curves: Horizontally Simple

Not all regions have areas that can be expressed in terms of $\int f(x) - g(x)\, dx$. The region in Figure 9 is called **horizontally simple** because any horizontal line through the region enters by intersecting one curve and leaves after intersecting the other. Note that this region is not vertically simple, so formula (1) does not apply. In order to compute the area of a horizontally simple region, we must convert the curves into functions of y instead of x. In other words, we want to work with functions of

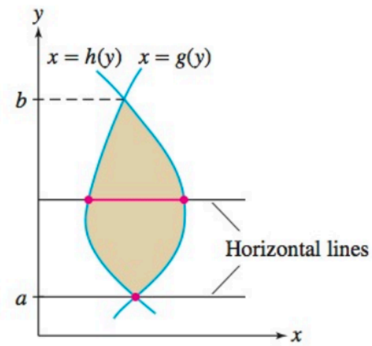

FIGURE 9 A horizontally simple region.

the form $x = h(y)$ rather than $y = h(x)$. If $x = h(y)$ and $x = g(y)$ are two curves such that $h(y) \geq g(y)$ for all y in some interval $[a, b]$ (so the graph of h lies to the **right** of g), then the area between the two curves is given by

$$\boxed{\text{Area between the graphs of } h \text{ and } g \quad = \quad \int_a^b [h(y) - g(y)]\, dy\,.} \tag{2}$$

Notice that the limits of integration are y-values. When finding the area of a horizontally simple region, we compute the integral of the **right minus left** functions, as opposed to "top minus bottom" for a vertically simple region.

Exercise 3: Returning to Example 1, compute the area enclosed by the curves $y = \sqrt{x}$ and $y = x$ by treating the region as horizontally simple and applying formula (2). Check that your answer agrees with the one obtained in Example 1.

Exercise 4: Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$ using formula (2). Sketch the region.

Extra Credit: Find the exact area (no decimals) of the shaded region in Figure 17.

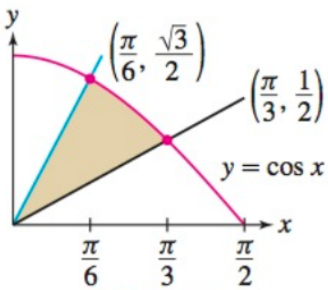

FIGURE 17