# MATH 134 Calculus 2 with FUndamentals

## Section 6.2: Volume, Density, and Average Value

In this section we highlight a few important applications of integration you are likely to encounter in your future courses. We learn how to find the volume of a three-dimensional solid by summing up areas of cross sections, how to find the average value of a function, and how to compute the total population given a radial density function.

### Average Value

Recall that the average of $n$ numbers $y_1, y_2, \ldots, y_n$ is given by $\dfrac{y_1 + y_2 + \cdots + y_n}{n}$.

How do we generalize this formula to find the average value of a function? For example, suppose we know the temperature in a room as a function of time. How do we calculate the average temperature over a particular time interval?

We begin by choosing $n$ points $x_1, x_2, \ldots, x_n$ in an interval $[a, b]$ and computing the average of their function values:

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

If we let $\Delta x = \frac{b-a}{n}$, then we find that $\frac{1}{n} = \frac{\Delta x}{b-a}$. Therefore

$$\text{Average Value} \approx \frac{\Delta x}{b-a}\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right) = \frac{1}{b-a}\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)\Delta x\,.$$

The right-hand side of the previous equation can be interpreted as a Riemann sum (e.g., a right-hand sum). Taking the limit as $n \to \infty$ (which means sampling the function at more and more $x$-values) leads to an integral definition for the average value of a function $f(x)$ on the interval $[a, b]$:

$$\boxed{\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)\,dx\,.} \tag{1}$$

**Example 1:** Find the average value of $f(x) = \sin x$ on the interval $[0, \pi]$.

**Answer:** Using the formula above, we compute the average value to be

$$\frac{1}{\pi - 0}\int_0^\pi \sin x\,dx = \frac{1}{\pi}\cdot -\cos x\big|_0^\pi = -\frac{1}{\pi}(\cos\pi - \cos 0) = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi} \approx 0.637\,.$$

One way to visualize the average value of a function is to interpret it as the average "height" (Figure 1).

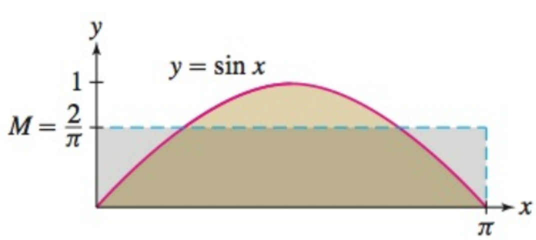



Figure 1: The average value of a function is the average height $M$ that makes the area under the graph equal to the area of the rectangle with height $M$ and base $b - a$.

**Exercise 1:** Find the average value of $f(x) = 4 - x^2$ on $[-2, 2]$. Illustrate your answer with a graph as in Figure 1.

**Exercise 2:** Find the average value of $g(t) = \sqrt{t}$ on $[1, 4]$.

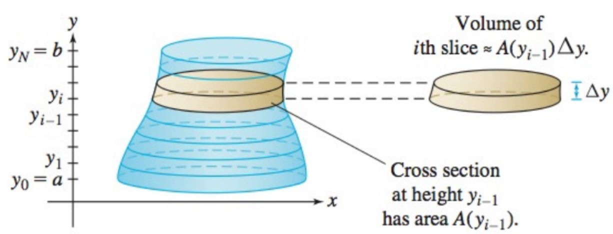
### Volume

How do we calculate the volume of a 3D solid? One technique is to divide the solid into thin cross sections and then add up the volumes of each cross section (see Figure 2). Suppose that there are $n$ cross sections each with the same width $\Delta y$. If we know the area of the cross section at a height of $y_i$ is given by $A(y_i)$, then the volume of that cross section is $A(y_i)\,\Delta y$ (see Figure 2). Thus, we have

$$\text{Volume} \approx A(y_0)\,\Delta y + A(y_1)\,\Delta y + \cdots + A(y_{n-1})\,\Delta y\,.$$

Once again, this can be interpreted as a Riemann sum. As $\Delta y \to 0$, the sum above becomes an integral over the range of heights in the $y$-direction:

$$\boxed{\text{Volume} = \int_a^b A(y)\,dy\,.} \tag{2}$$




DF **FIGURE 2** Divide the solid into thin horizontal slices. Each slice is nearly a right cylinder whose volume can be approximated as area times height.

Note that there is nothing special here about taking horizontal cross sections. This technique for finding the volume of a solid works perfectly well using vertical cross sections (perpendicular to the $x$-axis) between $x = a$ and $x = b$. In this case formula (2) becomes

$$\boxed{\text{Volume} = \int_a^b A(x)\,dx\,.} \tag{3}$$

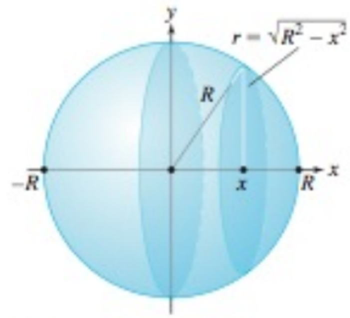



Figure 3: Finding the volume of a sphere of radius $R$ using circular cross sections perpendicular to the $x$-axis.

**Exercise 3:** Use Figure 3 and circular cross sections perpendicular to the $x$-axis to show that the volume of a sphere of radius $R$ is $V = \frac{4}{3}\pi R^3$.

**Exercise 4:** Find the volume of the solid whose base is the unit circle $x^2 + y^2 = 1$ with cross sections perpendicular to the $x$-axis equal to triangles whose height and base are equal.

**Hint:** Find the base of each triangle as a function of $x$.

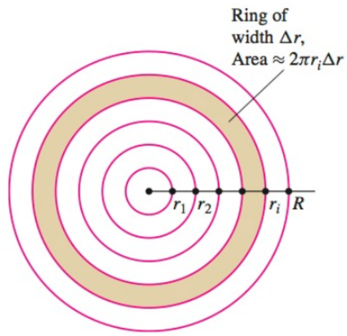



Figure 4: The area of a small ring of radius $r_i$ is approximately the outer circumference $2\pi r_i$ times the width $\Delta r$ (think of cutting the ring and stretching it out into a rectangle).

**Density: Computing Total Population**

Suppose that we want to calculate the population of a city (e.g., Boston) given a function $\rho(r)$ that measures the number of people living $r$ units from the center of the city. The function $\rho(r)$ is called a **radial density function** — think of concentric circles with a radius of $r$, where the population is roughly the same on a given circle. Typically $\rho(r)$ is a decreasing function since we expect there to be less people as we move further away from the center of the city.

The number of people living in a small circular ring of radius $r_i$ and width $\Delta r$ is approximately $2\pi r_i \cdot \Delta r \cdot \rho(r_i)$. This is the area of the ring times the population density on the ring (see Figure 4). To find the total population from the center of the city $r = 0$ to a particular radius $r = R$, we sum up the populations on each ring. As $\Delta r \to 0$, the sum becomes an integral. In other words, we have the following formula for computing total population:

$$\boxed{\text{Population within a radius } R \text{ of the center } = 2\pi \int_0^R r\rho(r)\, dr\,.} \tag{4}$$

**Exercise 5:** Suppose that Worcester's population has a radial density function of $\rho(r) = 12(2+r^2)^{-1}$, where $r$ is the distance (in miles) from the city center and $\rho$ is measured in thousands of people per square mile.

**(a)** Calculate the number of people living within 15 miles of the center of Worcester.

**(b)** How would you set up the integral to calculate the number of people living between 5 and 10 miles from the center of Worcester?