MATH 134 Calculus 2 with FUNdamentals

Section 7.7: Improper Integrals

Solutions

Exercises: Determine whether each integral converges or diverges. If the integral converges, find the value of the integral.

1. $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$

   Answer: The integral converges to $\pi/4$.

$$\begin{aligned}
\int_1^\infty \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{1+x^2}\,dx \\
&= \lim_{b\to\infty} \tan^{-1}(x)\,\Big|_1^b \\
&= \lim_{b\to\infty} \tan^{-1}(b) - \tan^{-1}(1) \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4},
\end{aligned}$$

   because the graph of $\tan^{-1} x$ goes to $\pi/2$ as $x \to \infty$ (horizontal asymptote). Thus the integral converges to $\pi/4$.

2. $\displaystyle\int_\pi^\infty \sin t\,dt$

   Answer: The integral diverges.

$$\begin{aligned}
\int_\pi^\infty \sin t\,dt &= \lim_{b\to\infty}\int_\pi^b \sin t\,dt \\
&= \lim_{b\to\infty} -\cos t\,\Big|_\pi^b \\
&= \lim_{b\to\infty} -\cos b + 1 \\
&= \text{Does Not Exist},
\end{aligned}$$

   because the graph of $-\cos t$ oscillates forever as $t \to \infty$. Thus the integral diverges.

3. $\displaystyle\int_0^\infty xe^{-4x}\,dx$

   Answer: The integral converges to $1/16$.

   To compute the integral, use integration by parts with $u = x$ and $dv = e^{-4x}\,dx$. Then $du = 1\,dx$ and $v = -\frac{1}{4}e^{-4x}$. Recall that $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$ and $\lim\limits_{b\to\infty} e^{-kb} = 0$ whenever $k > 0$ (exponential decay; the graph of e^{-kb} has a horizontal asymptote at $y = 0$). We have

$$\begin{aligned}
\int_0^{\infty} xe^{-4x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-4x}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{4}xe^{-4x}\Big|_0^b - \int_0^b -\frac{1}{4}e^{-4x}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{4}be^{-4b} - 0 - \frac{1}{16}e^{-4x}\Big|_0^b \\
&= \lim_{b\to\infty} -\frac{1}{4}be^{-4b} - \frac{1}{16}e^{-4b} + \frac{1}{16} \\
&= \frac{1}{16}\,.
\end{aligned}$$

The first limit can be computed using L'Hôpital's Rule:

$$\begin{aligned}
\lim_{b\to\infty} -\frac{1}{4}be^{-4b} &= \lim_{b\to\infty} -\frac{b}{4e^{4b}} \\
&= \lim_{b\to\infty} -\frac{1}{16e^{4b}} \\
&= 0\,.
\end{aligned}$$

4. $\displaystyle\int_0^2 \frac{1}{x}\,dx$

   Answer: The integral diverges.

$$\begin{aligned}
\int_0^2 \frac{1}{x}\,dx &= \lim_{b\to 0^+}\int_b^2 \frac{1}{x}\,dx \\
&= \lim_{b\to 0^+} \ln|x|\,\big|_b^2 \\
&= \lim_{b\to 0^+} \ln 2 - \ln b \\
&= \text{Does Not Exist},
\end{aligned}$$

because the graph of $\ln x$ has a vertical asymptote at $x = 0$ since $\lim\limits_{x\to 0+} \ln x = -\infty$. Therefore the integral diverges (area under the curve is infinite in the y-direction).

5. $\displaystyle\int_3^{\infty} \frac{1}{x^2-1}\,dx$

   Answer: The integral converges to $\frac{1}{2}\ln 2$.

   This integral can be computed using partial fractions. Notice that the denominator factors as $x^2 - 1 = (x+1)(x-1)$. To compute the partial fraction decomposition, we seek constants A and B such that

$$\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}\,.$$

Multiply both sides by the least common denominator $(x+1)(x-1)$:

$$(x+1)(x-1)\left(\frac{1}{x^2-1}\right) = (x+1)(x-1)\left(\frac{A}{x+1}+\frac{B}{x-1}\right).$$

After cancelling, this gives

$$1 = A(x-1)+B(x+1).$$

To find A and B, plug in the roots of the original denominator: $x=1$ and $x=-1$. Plugging in $x=1$ gives $1 = A\cdot 0+B\cdot 2$, which implies $B=1/2$. Plugging in $x=-1$ gives $1 = A\cdot(-2)+B\cdot 0$, which implies $A=-1/2$.

To compute the integral, we break the fraction into two pieces:

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{-1/2}{x+1}+\frac{1/2}{x-1}\,dx \\ &= -\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1| \\ &= \frac{1}{2}(\ln|x-1|-\ln|x+1|) \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|.\end{aligned}$$

The last step above, which follows from the property $\ln a - \ln b = \ln(a/b)$, is crucial for what comes next.

$$\begin{aligned}\int_3^\infty \frac{1}{x^2-1}\,dx &= \lim_{b\to\infty}\int_3^b \frac{1}{x^2-1}\,dx \\ &= \lim_{b\to\infty}\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\Bigg|_3^b \\ &= \lim_{b\to\infty}\frac{1}{2}\ln\left|\frac{b-1}{b+1}\right| - \frac{1}{2}\ln\left(\frac{1}{2}\right) \\ &= 0-\frac{1}{2}\ln\left(\frac{1}{2}\right) \\ &= \frac{1}{2}\ln 2,\end{aligned}$$

since $\ln a^b = b\ln a$ and $1/2 = 2^{-1}$. The limit above equals $\frac{1}{2}\ln 1 = 0$ using L'Hôpital's Rule (focus on the limit inside the ln function). Thus the integral converges to $\frac{1}{2}\ln 2$.

6. $\displaystyle\int_0^4 \frac{1}{\sqrt{16-x^2}}\,dx$

Answer: The integral converges to $\pi/2$.

Notice that the "bad" point is $x=4$ since at this value, the denominator of the integrand becomes 0. The integral can be computed using the trig substitution $x = 4\sin\theta$. Then $dx =$

$4\cos\theta\, d\theta$ and $16 - x^2 = 16 - 16\sin^2\theta = 16(1-\sin^2\theta) = 16\cos^2\theta$. We find

$$\begin{aligned}
\int_0^4 \frac{1}{\sqrt{16-x^2}}\,dx &= \lim_{b\to 4}\int_0^b \frac{1}{\sqrt{16-x^2}}\,dx \\
&= \lim_{b\to 4}\int_0^b \frac{1}{\sqrt{16\cos^2\theta}}\cdot 4\cos\theta\, d\theta \\
&= \lim_{b\to 4}\int_0^b 1\, d\theta \\
&= \lim_{b\to 4}\ \theta \qquad \text{(take note of the correct variable here)} \\
&= \lim_{b\to 4}\ \sin^{-1}\left(\frac{x}{4}\right)\Big|_0^b \\
&= \lim_{b\to 4}\ \sin^{-1}(b/4) - \sin^{-1}(0) \\
&= \sin^{-1}(1) - \sin^{-1}(0) \\
&= \frac{\pi}{2}\,.
\end{aligned}$$