MATH 134 Calculus 2 with FUNdamentals

Section 7.7: Improper Integrals

This section covers integrals that have ∞ or $-\infty$ (or both) as a limit of integration, as well as integrals where the integrand has a vertical asymptote at one of the limits of integration (or between them). These kinds of integrals are called **improper integrals**.

Definition: An **improper integral** is a definite integral where either

(a) a limit of integration is ∞ or $-\infty$, or

(b) the integrand approaches ∞ or $-\infty$ at a limit of integration or somewhere in between the limits of integration.

The following are all examples of improper integrals. The first two are of type **(a)** in the above definition, while the second two are of type **(b)**.

$$\int_0^\infty e^{-2x}\,dx, \quad \int_{-\infty}^1 \frac{1}{x^2+1}\,dx, \quad \int_0^1 \frac{1}{x}\,dx, \quad \int_{-2}^1 \frac{1}{x}\,dx\,.$$

In terms of area, improper integrals of type **(a)** are used to determine whether the area under a curve is finite, even if one of the limits of integration is infinite. Improper integrals of type **(b)** ask the same question except now the area is in the vertical direction (up against a vertical asymptote).

Here is the general approach for evaluating improper integrals:

1. Determine that the definite integral is improper.
2. Replace the "bad" limit of integration with a variable b.
3. Find the antiderivative of the integrand and plug in b.
4. Take the limit of the result as b approaches the "bad" value. If this limit exists and is finite, we say the integral **converges**, if the limit does not exist, we say the integral **diverges**.

Example 1: A convergent integral

Consider the integral $\int_0^\infty e^{-2x}\,dx$. Notice that the integrand e^{-2x} approaches 0 as $x \to \infty$ (a decaying exponential). However, is the area under the curve finite or infinite as $x \to \infty$? Somewhat surprisingly, the area is indeed finite (it equals 1/2). Even though $x \to \infty$, the integrand approaches 0 so fast that the area under the curve is finite. To see this, we compute

$$\begin{aligned}
\int_0^\infty e^{-2x}\,dx &= \lim_{b\to\infty} \int_0^b e^{-2x}\,dx \\
&= \lim_{b\to\infty} -\frac{1}{2}e^{-2x}\Big|_0^b \\
&= \lim_{b\to\infty} -\frac{1}{2}e^{-2b} + \frac{1}{2} \\
&= \frac{1}{2}\,.
\end{aligned}$$

Thus the integral **converges** to 1/2.

Example 2: A divergent integral

Consider the integral $\int_1^\infty \frac{1}{x}\,dx$. As with Example 1, the integrand approaches 0 as $x \to \infty$. However, this time the area under the curve is infinite. The function $f(x) = 1/x$ does not approach 0 fast enough to have finite area. In this case we have

$$\begin{aligned} \int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \ln|x|\,\Big|_1^b \\ &= \lim_{b\to\infty} \ln b - 0 \\ &= \infty\,, \end{aligned}$$

because the graph of $\ln x$ goes to ∞ (slowly) as $x \to \infty$. Thus the integral **diverges.**

Example 3: A convergent integral of type (b)

Consider the integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. This integral is improper because the integrand is undefined at $x = 0$ (vertical asymptote). However, the area under the curve is finite despite the fact that the integrand is going to ∞ as $x \to 0^+$. In this case, the integrand goes to ∞ *slow* enough for the area to be finite. To see this, we compute

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{b\to 0^+} \int_b^1 \frac{1}{\sqrt{x}}\,dx \\ &= \lim_{b\to 0^+} 2x^{1/2}\,\Big|_b^1 \\ &= \lim_{b\to 0^+} 2 - 2\sqrt{b} \\ &= 2\,. \end{aligned}$$

Thus the integral converges to 2.

Exercises: Determine whether each integral converges or diverges. If the integral converges, find the value of the integral.

1. $\int_1^\infty \frac{1}{1+x^2}\,dx$

2. $\int_\pi^\infty \sin t\,dt$

3. $\int_0^\infty xe^{-4x}\, dx$

4. $\int_0^2 \frac{1}{x}\, dx$

5. $\int_3^\infty \frac{1}{x^2 - 1}\, dx$

6. $\int_0^4 \frac{1}{\sqrt{16 - x^2}}\, dx$