MATH 134 Calculus 2 with FUNdamentals

Section 7.1: Integration by Parts

Chapter 7 covers various types of integration techniques. In the first section we learn integration by parts, which is essentially the **product rule** for integrals. It is derived by starting with the product rule for differentiation, moving one term to the other side of the equation, and then integrating both sides.

$$\begin{aligned}(u\,v)' = u'v + uv' \quad &\Longrightarrow \quad uv' = (u\,v)' - u'v \\ &\Longrightarrow \quad \int uv' = \int (u\,v)' - \int u'v \\ &\Longrightarrow \quad \int uv' = u\,v - \int u'v\end{aligned}$$

The last equation is typically written as

$$\boxed{\int u\,dv \;=\; u\,v - \int v\,du\,.} \tag{1}$$

Equation (1) is called the **integration by parts** formula. It is particularly useful for integrals of products when there is no simplification or u-substitution available to compute the integral. The key to applying integration by parts is to determine which function represents u and which function represents dv (the derivative of v).

Example 1: Compute $\int xe^x\,dx$ using integration by parts.

Answer: We choose $u = x$ and $dv = e^x\,dx$. Then we have $du = 1\,dx$ and $v = e^x$. Note that du is computed by taking the derivative of u, while v is obtained by finding an **antiderivative** of dv (we always take $c = 0$ here since that is easiest). Applying formula (1), we have

$$\int xe^x\,dx \;=\; xe^x - \int e^x\,dx \;=\; xe^x - e^x + c \;=\; e^x(x-1) + c\,.$$

Tips for using integration by parts:

- Choose u so that du is simpler than u (e.g., power functions x^n or $\ln x$).
- Be sure you can integrate dv to find v. If not, you need to choose a different dv.
- Integration by parts works if the new integral obtained is **easier** than the original. Sometimes you need to do the technique twice to obtain an integral that is computable. Other times you might do integration by parts twice and get back to the original integral. This is actually ok, as then you can do some algebra to solve for the original integral (see Exercise 6).

Exercises

1. Use integration by parts to compute $\int x\sin x\,dx$. Let $u = x$ and $dv = \sin x\,dx$.

2. Compute $\displaystyle\int \ln x \, dx$ by setting $u = \ln x$ and $dv = 1\, dx$.

3. Evaluate $\displaystyle\int x^3 \ln x \, dx$.

4. Evaluate $\displaystyle\int x^2 \sin(2x) \, dx$.

5. Evaluate $\displaystyle\int_0^1 \tan^{-1} x \, dx$.

6. Evaluate $\displaystyle\int e^{3x} \cos x \, dx$.

   Hint: Let $I = \displaystyle\int e^{3x} \cos x \, dx$. Do integration by parts twice and then solve for I.