MATH 134 Calculus 2 with FUNdamentals

Section 7.6: Strategies for Integration

Solutions

Exercises: Evaluate each of the following integrals by choosing an appropriate technique(s) of integration.

1. $\displaystyle\int x\sqrt{x^2+1}\,dx$

   Answer: This is easiest to compute using a u-substitution. Let $u = x^2+1$. Then $du = 2x\,dx$ or $du/2 = x\,dx$ and we have

$$\int x\sqrt{x^2+1}\,dx \;=\; \int \sqrt{u}\cdot\frac{du}{2} \;=\; \frac{1}{2}\int u^{1/2}\,du \;=\; \frac{1}{2}\cdot\frac{2}{3}u^{3/2}+c \;=\; \frac{1}{3}\left(x^2+1\right)^{3/2}+c\,.$$

2. $\displaystyle\int x^3\sqrt{x^2+1}\,dx$ **Hint:** Use the u-sub $u = x^2+1$ and write $x^3 = x^2\cdot x$.

   Answer: This can also be computed using a u-substitution, which is easier than doing trig sub. Let $u = x^2+1$. Then $du = 2x\,dx$ or $du/2 = x\,dx$. Moreover, $x^2 = u-1$. If we split the x^3 term into $x\cdot x^2$, we can evaluate the integral using the power rule. We find

$$\begin{aligned}
\int x^3\sqrt{x^2+1}\,dx &= \int x\cdot x^2\sqrt{x^2+1}\,dx \\
&= \int (u-1)\sqrt{u}\cdot\frac{du}{2} \\
&= \frac{1}{2}\int (u-1)u^{1/2}\,du \\
&= \frac{1}{2}\int u^{3/2}-u^{1/2}\,du \\
&= \frac{1}{2}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\,du \\
&= \frac{1}{5}(x^2+1)^{5/2}-\frac{1}{3}(x^2+1)^{3/2}+c\,.
\end{aligned}$$

3. $\displaystyle\int t^2e^{3t}\,dt$

   Answer: This integral can be evaluated using integration by parts twice. First, let $u = t^2$ and $dv = e^{3t}\,dt$. Then $du = 2t\,dt$ and $v = \frac{1}{3}e^{3t}$. We find

$$\int t^2e^{3t}\,dt \;=\; t^2\cdot\frac{1}{3}e^{3t}-\int\frac{1}{3}e^{3t}\cdot 2t\,dt \;=\; \frac{1}{3}t^2e^{3t}-\frac{2}{3}\int te^{3t}\,dt\,.$$

The remaining integral can be done using integration by parts again. Let $u = t$ and $dv = e^{3t}\,dt$. Then $du = dt$ and $v = \frac{1}{3}e^{3t}$. We compute

$$\begin{aligned}
\int t^2 e^{3t}\,dt &= \frac{1}{3}t^2e^{3t} - \frac{2}{3}\int te^{3t}\,dt \\
&= \frac{1}{3}t^2e^{3t} - \frac{2}{3}\left(t\cdot\frac{1}{3}e^{3t} - \int \frac{1}{3}e^{3t}\,dt\right) \\
&= \frac{1}{3}t^2e^{3t} - \frac{2}{3}\left(\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + c\right) \\
&= \frac{1}{3}t^2e^{3t} - \frac{2}{9}te^{3t} + \frac{2}{27}e^{3t} + c \\
&= \frac{1}{27}e^{3t}\left(9t^2 - 6t + 2\right) + c\,.
\end{aligned}$$

4. $\displaystyle\int_0^{\pi/4} \sin^4\theta\,d\theta$

Answer: To evaluate this integral, we apply the formula $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$. Being careful to FOIL the integrand, we find

$$\int_0^{\pi/4}\sin^4\theta\,d\theta = \int_0^{\pi/4}\left(\frac{1}{2}(1-\cos(2\theta))\right)^2 d\theta = \frac{1}{4}\int_0^{\pi/4} 1 - 2\cos(2\theta) + \cos^2(2\theta)\,d\theta\,.$$

Next we use the formula $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$, except we replace θ by 2θ. We find

$$\begin{aligned}
\frac{1}{4}\int_0^{\pi/4} 1 - 2\cos(2\theta) + \cos^2(2\theta)\,d\theta &= \frac{1}{4}\int_0^{\pi/4} 1 - 2\cos(2\theta) + \frac{1}{2}(1+\cos(4\theta))\,d\theta \\
&= \frac{1}{4}\int_0^{\pi/4} \frac{3}{2} - 2\cos(2\theta) + \frac{1}{2}\cos(4\theta)\,d\theta \\
&= \frac{1}{4}\left(\frac{3}{2}\theta - \sin(2\theta) + \frac{1}{8}\sin(4\theta)\Big|_0^{\pi/4}\right) \\
&= \frac{1}{4}\left(\frac{3}{2}\cdot\frac{\pi}{4} - \sin(\pi/2) + \frac{1}{8}\sin(\pi) - 0\right) \\
&= \frac{3\pi}{32} - \frac{1}{4}\,.
\end{aligned}$$

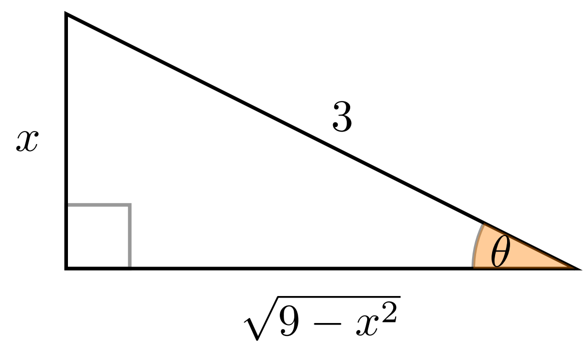
5. $\displaystyle\int \frac{x^2}{(9-x^2)^{3/2}}\,dx$ **Hint:** After making a trig substitution, use the identity $\tan^2\theta + 1 = \sec^2\theta$ to evaluate the integral.

Answer: This problem can be computed using the trig substitution $x = 3\sin\theta$. Then $dx =$

$3\cos\theta\, d\theta$ and $9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta$. We find

$$\begin{aligned}
\int \frac{x^2}{(9-x^2)^{3/2}}\, dx &= \int \frac{9\sin^2\theta}{(9\cos^2\theta)^{3/2}} \cdot 3\cos\theta\, d\theta \\
&= \int \frac{27\sin^2\theta\cos\theta}{27\cos^3\theta}\, d\theta \\
&= \int \tan^2\theta\, d\theta \\
&= \int \sec^2\theta - 1\, d\theta \\
&= \tan\theta - \theta + c \\
&= \frac{x}{\sqrt{9-x^2}} - \sin^{-1}\left(\frac{x}{3}\right) + c\,.
\end{aligned}$$

The final step comes from using SOH-CAH-TOA, $\sin\theta = x/3$, and the Pythagorean Theorem (see figure below).



6. $\displaystyle\int \frac{24}{x^2-9}\, dx$

Answer: This integral can be computed using partial fractions. Notice that the denominator factors as $x^2 - 9 = (x+3)(x-3)$. To compute the partial fraction decomposition, we seek constants A and B such that

$$\frac{24}{x^2-9} = \frac{A}{x+3} + \frac{B}{x-3}\,.$$

Multiply both sides by the least common denominator $(x+3)(x-3)$:

$$(x+3)(x-3)\left(\frac{24}{x^2-9}\right) = (x+3)(x-3)\left(\frac{A}{x+3} + \frac{B}{x-3}\right).$$

After cancelling, this gives

$$24 = A(x-3) + B(x+3).$$

To find A and B, plug in the roots of the original denominator: $x = 3$ and $x = -3$. Plugging in $x = 3$ gives $24 = A\cdot 0 + B\cdot 6$, which implies $B = 4$. Plugging in $x = -3$ gives $24 = A\cdot(-6) + B\cdot 0$, which implies $A = -4$.

To compute the integral, we break the fraction into two pieces:

$$\begin{aligned}
\int \frac{24}{x^2-9}\,dx &= \int \frac{-4}{x+3} + \frac{4}{x-3}\,dx \\
&= -4\ln|x+3| + 4\ln|x-3| + c \\
&= 4(\ln|x-3| - \ln|x+3|) + c \\
&= 4\ln\left|\frac{x-3}{x+3}\right| + c\,.
\end{aligned}$$