MATH 134 Calculus 2 with FUNdamentals

Section 7.8: Probability and Integration

Solutions

1. Find the value of C that makes $p(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{C}{(x+2)^2} & \text{if } x \geq 0 \end{cases}$ a probability density function. Then compute $P(0 \leq x \leq 1)$ and $P(x \geq 1)$.

   Answer: Since $p(x) = 0$ for $-\infty < x < 0$, we need to solve $\displaystyle\int_0^\infty \frac{C}{(x+2)^2}\,dx = 1$ for C. We have

$$\begin{aligned}
\int_0^\infty \frac{C}{(x+2)^2}\,dx &= \lim_{b\to\infty} \int_0^b \frac{C}{(x+2)^2}\,dx \\
&= \lim_{b\to\infty} C \int_0^b (x+2)^{-2}\,dx \\
&= C \lim_{b\to\infty} -(x+2)^{-1}\Big|_0^b \qquad (u\text{-sub with } u = x+2) \\
&= C\left(\lim_{b\to\infty} -(b+2)^{-1} + \frac{1}{2}\right) \\
&= C\left(\lim_{b\to\infty} -\frac{1}{b+2} + \frac{1}{2}\right) \\
&= \frac{C}{2}.
\end{aligned}$$

   Thus we must solve $\frac{C}{2} = 1$, which gives $C = 2$.

   Now that we have the correct value of C, we compute

$$P(0 \leq x \leq 1) = \int_0^1 2(x+2)^{-2}\,dx = -2(x+2)^{-1}\Big|_0^1 = -\frac{2}{3} + 1 = \frac{1}{3}.$$

   Finally, since $P(0 \leq x \leq 1) = \frac{1}{3}$ and $P(x \geq 0) = 1$ (by definition), it follows by linearity that

$$P(x \geq 1) = 1 - \frac{1}{3} = \frac{2}{3}.$$

   Note: You will get the same result by integrating the PDF from 1 to ∞, but using linearity is a much quicker (and easier) approach.

2. Show that $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ke^{-kx} & \text{if } x \geq 0 \end{cases}$ is a probability density function for any constant $k > 0$.

   This PDF is known as the **exponential density function**.

   Answer: First notice that $f(x) \geq 0$ is true since $k > 0$. The key item to check is property (ii), that $\displaystyle\int_{-\infty}^\infty f(x)\,dx = 1$. Since $f(x) = 0$ for $-\infty < x < 0$, we must show that $\displaystyle\int_0^\infty ke^{-kx}\,dx = 1$

for any choice of $k > 0$. We have

$$\begin{aligned}
\int_0^\infty ke^{-kx}\,dx &= \lim_{b\to\infty}\int_0^b ke^{-kx}\,dx \\
&= \lim_{b\to\infty} -e^{-kx}\Big|_0^b \\
&= \lim_{b\to\infty} -e^{-kb}+1 \\
&= \lim_{b\to\infty} -\frac{1}{e^{kb}}+1 \\
&= 1,
\end{aligned}$$

since $k > 0$. Note that if $k < 0$, the limit above would approach $-\infty$ and so the integral would diverge. This shows that $f(x)$ is a PDF.

3. Suppose that the probability a telephone call made in the US lasts between a and b minutes is modeled by the exponential density function with $k = 1/4$.

   a) What is the probability that a call lasts between 2 and 3 minutes?

   b) What is the probability that a call lasts over an hour?

   Answer: We use the PDF from the previous problem replacing k by $1/4$. For part **a)**, we compute

$$P(2 \le x \le 3) = \int_2^3 \frac{1}{4}e^{-\frac{1}{4}x}\,dx = -e^{-\frac{1}{4}x}\Big|_2^3 = -e^{-\frac{3}{4}} + e^{-\frac{1}{2}} \approx 0.134 = 13.4\%.$$

   For part **b)**, we compute

$$\begin{aligned}
P(x > 60) &= \int_{60}^\infty \frac{1}{4}e^{-\frac{1}{4}x}\,dx \\
&= \lim_{b\to\infty}\int_{60}^b \frac{1}{4}e^{-\frac{1}{4}x}\,dx \\
&= \lim_{b\to\infty} -e^{-\frac{1}{4}x}\Big|_{60}^b \\
&= \lim_{b\to\infty} -\frac{1}{e^{\frac{1}{4}b}} + e^{-15} \\
&= e^{-15} \\
&\approx 0.000000306 = 0\%.
\end{aligned}$$

4. Show that the mean of the exponential density function is $1/k$.

   Answer: We must show that $\int_{-\infty}^\infty xf(x)\,dx = 1/k$ where $f(x)$ is the exponential density function defined in Exercise 2. Since $f(x) = 0$ when $x < 0$, this reduces to showing $\int_0^\infty xke^{-kx}\,dx = 1/k$.

The integral can be computed using integration by parts with $u = x$ and $dv = ke^{-kx}\,dx$. Then $du = dx$ and $v = -e^{-kx}$. We compute

$$\begin{aligned}
\int_0^\infty xke^{-kx}\,dx &= \lim_{b\to\infty} \int_0^b xke^{-kx}\,dx \\
&= \lim_{b\to\infty} -xe^{-kx}\Big|_0^b - \int_0^b -e^{-kx}\,dx \\
&= \lim_{b\to\infty} -be^{-kb} - 0 - \frac{1}{k}e^{-kx}\Big|_0^b \\
&= \lim_{b\to\infty} -\frac{b}{e^{kb}} - \frac{1}{k}e^{-kb} + \frac{1}{k} \\
&= \lim_{b\to\infty} -\frac{1}{ke^{kb}} - \frac{1}{ke^{kb}} + \frac{1}{k} \\
&= \frac{1}{k},
\end{aligned}$$

since the first two terms above both approach 0 as $b \to \infty$ (for the first one, use L'Höpital's Rule). Thus, the mean of the exponential density function is $1/k$.

5. Show that $f(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2} & \text{if } -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$ is a probability density function, and then calculate its mean. **Hint:** Draw a graph of f and interpret the integrals in terms of area.

Answer: To see that $f(x)$ is a PDF, we need to check that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Since the function is zero for $x < -2$ and $x > 2$, we must check that

$$\int_{-2}^{2} \frac{1}{2\pi}\sqrt{4-x^2}\,dx = 1 \quad \text{or} \quad \frac{1}{2\pi}\int_{-2}^{2} \sqrt{4-x^2}\,dx = 1\,.$$

Although we *could* use trig sub with $x = 2\sin\theta$ to compute the integral, the easiest approach is to recognize that $y = \sqrt{4-x^2}$ is the top half of the graph of a circle of radius 2 ($x^2+y^2=4$). Since we are integrating from -2 to 2, the integral is equivalent to the area of the semi-circle, which is $\frac{1}{2}\pi(2)^2 = 2\pi$. Thus, the 2π and $\frac{1}{2\pi}$ cancel out, giving a value of 1, as desired.

Next, we compute the mean μ, using the formula $\mu = \int_{-\infty}^{\infty} xf(x)\,dx$. Again, since the function is zero for $x < -2$ and $x > 2$, we only need to integrate from -2 to 2. The mean is

$$\begin{aligned}
\int_{-2}^{2} x \cdot \frac{1}{2\pi}\sqrt{4-x^2}\,dx &= \frac{1}{2\pi}\int_{-2}^{2} x\sqrt{4-x^2}\,dx \\
&= \frac{1}{2\pi}\int_0^0 \sqrt{u}\cdot\frac{du}{-2} \qquad (u\text{-sub with } u = 4-x^2 \text{ and } du = -2x\,dx) \\
&= -\frac{1}{4\pi}\int_0^0 \sqrt{u}\,du \\
&= 0\,.
\end{aligned}$$

This makes intuitive sense as the graph of the PDF (a semi-circle) is symmetric about $x = 0$.