# MATH 134 Calculus 2 with FUNdamentals

## Section 7.3: Trigonometric Substitution

This section focuses on integrals involving expressions of the form $\sqrt{a^2 - x^2}$ or $\sqrt{x^2 + a^2}$, where $a$ is a constant. For these types of integrals, the key is to make a **trig substitution** that converts the integral into a simpler form. Then, after computing the integral in the new variable (usually $\theta$), we use a right triangle and SOH-CAH-TOA to convert back into the original variable.

**Useful Trig Identities:**

$$\cos^2\theta + \sin^2\theta = 1 \tag{1}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{2}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{3}$$

Note that identity (2) above can be derived from (1) by dividing both sides of the equation by $\cos^2\theta$.

The general technique of trig substitution is:

- for integrals containing $\sqrt{a^2 - x^2}$, use the substitution $x = a\sin\theta$,
- for integrals with $\sqrt{x^2 + a^2}$, or $x^2 + a^2$ in the denominator, use the substitution $x = a\tan\theta$.

### 1) Integrals involving $\sqrt{a^2 - x^2}$

**Example 1:**

Compute $\displaystyle\int \sqrt{9 - x^2}\,dx$ using the substitution $x = 3\sin\theta$.

**Answer:** Letting $x = 3\sin\theta$, we have $dx = 3\cos\theta\,d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9(1 - \sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta.$$

Thus, the integral transforms to

$$\begin{aligned}
\int 3\cos\theta \cdot 3\cos\theta\,d\theta &= 9\int \cos^2\theta\,d\theta \\
&= 9\int \frac{1}{2}\left(1 + \cos(2\theta)\right)\,d\theta \qquad \text{(identity from Section 7.2)} \\
&= \frac{9}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + c \\
&= \frac{9}{2}(\theta + \sin\theta\cos\theta) + c \qquad \text{(using formula (3))}.
\end{aligned}$$

To finish the problem, we return to the original variable $x$. We have $\sin\theta = x/3$, so $\theta = \sin^{-1}(x/3)$. Using a right triangle with one angle equal to $\theta$, we find $\cos\theta = \frac{1}{3}\sqrt{9 - x^2}$. Hence, the solution is

$$\frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{3}\cdot\frac{\sqrt{9 - x^2}}{3}\right) = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}x\sqrt{9 - x^2} + c\,.$$

**2) Integrals involving $\sqrt{x^2+a^2}$ or $x^2+a^2$**

**Example 2:** Evaluate $\displaystyle\int_5^{5\sqrt{3}} \frac{1}{x(x^2+25)}\,dx$ using the substitution $x = 5\tan\theta$.

**Answer:** Letting $x = 5\tan\theta$, we have $dx = 5\sec^2\theta\,d\theta$ and

$$x^2+25 \;=\; 25\tan^2\theta+25 \;=\; 25(\tan^2\theta+1) \;=\; 25\sec^2\theta.$$

Moreover, since $\tan\theta = x/5$, we have $x = 5$ implies $\tan\theta = 1$, which means $\theta = \pi/4$, and $x = 5\sqrt{3}$ implies $\tan\theta = \sqrt{3}$ yielding $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$. Thus, the integral transforms to

$$\begin{aligned}
\int_{\pi/4}^{\pi/3} \frac{1}{5\tan\theta(25\sec^2\theta)}\cdot 5\sec^2\theta\,d\theta &= \int_{\pi/4}^{\pi/3}\frac{1}{25\tan\theta}\,d\theta \\
&= \frac{1}{25}\int_{\pi/4}^{\pi/3}\frac{\cos\theta}{\sin\theta}\,d\theta \qquad (u\text{-sub with } u=\sin\theta) \\
&= \frac{1}{25}\ln|\sin\theta|\Big|_{\pi/4}^{\pi/3} \\
&= \frac{1}{25}\left(\ln|\sin(\pi/3)| - \ln|\sin(\pi/4)|\right) \\
&= \frac{1}{25}\Big(\ln(\sqrt{3}/2) - \ln(\sqrt{2}/2)\Big) \\
&= \frac{1}{25}\ln(\sqrt{3/2}\,) \qquad (\ln a - \ln b = \ln(a/b)\ ).
\end{aligned}$$

**Exercises:** Complete the following on a **separate** piece(s) of paper.

1. Evaluate $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$.

   Check your answer by interpreting the integral as the area under the curve.

2. Evaluate $\displaystyle\int \frac{1}{x(x^2+4)}\,dx$ using the substitution $x = 2\tan\theta$.

3. Evaluate $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$.

4. Evaluate $\displaystyle\int_{\sqrt{3}}^3 \frac{1}{x^2\sqrt{x^2+9}}\,dx$.