MATH 134 Calculus 2 with FUNdamentals

Section 9.1: Solving Differential Equations

Solutions

Exercise 1: Check that $y = A\sin 2t + B\cos 2t$ satisfies the ODE $y'' + 4y = 0$ for any constants A and B. Here y'' is the second derivative of y with respect to t.

Answer: To check that the given function is a solution to the ODE, we simply plug it into the equation and show that the equation is satisfied. Using the chain rule, we have $y' = A\cos(2t)\cdot 2 - B\sin(2t)\cdot 2 = 2A\cos(2t) - 2B\sin(2t)$. Then,

$$y'' = -2A\sin(2t)\cdot 2 - 2B\cos(2t)\cdot 2 = -4A\sin(2t) - 4B\cos(2t)$$

and thus

$$\begin{aligned} y'' + 4y &= -4A\sin(2t) - 4B\cos(2t) + 4(A\sin 2t + B\cos 2t) \\ &= -4A\sin(2t) - 4B\cos(2t) + 4A\sin 2t + 4B\cos 2t \\ &= 0 \end{aligned}$$

as desired.

Exercises:

2. Use the Separation of Variables technique to find the general solution to $\dfrac{dy}{dt} = ry$, where r is a constant. Where have we seen this formula before?

   Answer: We begin by moving the terms with y to the left-hand side of the equation and those with t to the right. Keep in mind that r is a constant.

   $$\frac{dy}{dt} = ry \implies \frac{1}{y}\,dy = r\,dt\,.$$

   Next we integrate both sides, integrating on the left-hand side with respect to y and integrating on the right-hand side with respect to t. This gives

   $$\int \frac{1}{y}\,dy = \int r\,dt \implies \ln|y| = rt + c\,.$$

   Notice that we only have one integration constant c on the right-hand side. If we had a constant on the left-hand side as well (say d), we would have moved it over to the right-hand side and combined it with c (replacing $c - d$ with just c). Now we solve for y by raising both sides to the base e:

   $$e^{\ln|y|} = e^{rt+c} = e^{rt}\cdot e^{c} = ce^{rt} \implies |y| = ce^{rt}\,,$$

   where we have replaced the constant e^c with just c (they are both arbitrary constants so we opt for the simplest choice c). Thus, $y = \pm ce^{rt}$, which can be condensed to just $y = ce^{rt}$, with $c \in \mathbb{R}$ an arbitrary constant. The general solution is $y = ce^{rt}$ (check that it satisfies the ODE).

   This problem models continuously compounded interest, where r is the interest rate. If $y(t)$ represents the amount of money in an account, then the ODE can be interpreted as the rate the account is changing is proportional to the amount of money in the account. In other words, the more money in your account (or the larger the interest rate r is), the more your account stands to grow.

3. Use the Separation of Variables technique to find the general solution to $y' = y^2 \sin(4x)$. Then find the particular solution satisfying $y(0) = 1$.

Answer: First we separate the variables:

$$\frac{dy}{dt} = y^2 \sin(4x) \quad \Longrightarrow \quad \frac{1}{y^2}\, dy = \sin(4x)\, dx\,.$$

Next we integrate both sides and solve for y:

$$\begin{aligned} \int y^{-2}\, dy = \int \sin(4x)\, dx \quad &\Longrightarrow \quad -y^{-1} = -\frac{1}{4}\cos(4x) + c \\ &\Longrightarrow \quad \frac{1}{y} = \frac{1}{4}\cos(4x) + c \\ &\Longrightarrow \quad y = \frac{1}{\frac{1}{4}\cos(4x) + c}\,. \end{aligned}$$

Here we use the trick $\frac{1}{y} = f(x) \implies y = \frac{1}{f(x)}$ in the final step.

To find the particular solution satisfying $y(0) = 1$, we plug in $x = 0$ and $y = 1$ into the general solution we just found and solve for the constant c. This gives

$$1 = \frac{1}{\frac{1}{4}\cos(0) + c} \quad \Longrightarrow \quad 1 = \frac{1}{\frac{1}{4} + c} \quad \Longrightarrow \quad \frac{1}{4} + c = 1 \quad \Longrightarrow \quad c = \frac{3}{4}.$$

Therefore, the particular solution we seek is $y = \dfrac{1}{\frac{1}{4}\cos(4x) + \frac{3}{4}}$.

4. Find the solution to $y' = (1 - t^2)(1 + y^2)$ satisfying the initial condition $y(0) = -1$.

Answer: First we separate the variables:

$$\frac{dy}{dt} = (1 - t^2)(1 + y^2) \quad \Longrightarrow \quad \frac{1}{1 + y^2}\, dy = 1 - t^2\, dt\,.$$

Next we integrate both sides and solve for y:

$$\begin{aligned} \int \frac{1}{1 + y^2}\, dy = \int 1 - t^2\, dt \quad &\Longrightarrow \quad \tan^{-1}(y) = t - \frac{t^3}{3} + c \\ &\Longrightarrow \quad y = \tan\left(t - \frac{t^3}{3} + c\right)\,. \end{aligned}$$

To find the particular solution satisfying $y(0) = -1$, we plug in $t = 0$ and $y = -1$ into the general solution we just found and solve for the constant c. This gives

$$-1 = \tan(0 + c) \quad \Longrightarrow \quad -1 = \tan c \quad \Longrightarrow \quad c = \tan^{-1}(-1) \quad \Longrightarrow \quad c = -\frac{\pi}{4}.$$

Therefore, the particular solution we seek is $y = \tan\left(t - \dfrac{t^3}{3} - \dfrac{\pi}{4}\right)$.

5. Find the solution to $\sqrt{1-x^2}\, y' = 2x\sqrt{y}\,, \;\; y(0) = 9$.

Answer: First we rewrite the equation as $\dfrac{dy}{dx} \;=\; \dfrac{2x\sqrt{y}}{\sqrt{1-x^2}}$.

Then we separate the variables:

$$\frac{dy}{dx} \;=\; \frac{2x\sqrt{y}}{\sqrt{1-x^2}} \qquad \Longrightarrow \qquad \frac{1}{\sqrt{y}}\, dy \;=\; \frac{2x}{\sqrt{1-x^2}}\, dx\,.$$

Next we integrate both sides and solve for y. The integral with respect to x is a u-sub with $u = 1-x^2$ and $du = -2x\,dx$.

$$\begin{aligned}
\int y^{-1/2}\, dy \;=\; \int \frac{2x}{\sqrt{1-x^2}}\, dx \;\Longrightarrow\;\; 2y^{1/2} \;&=\; -\int \frac{1}{\sqrt{u}}\, du \\
\Longrightarrow\;\; 2y^{1/2} \;&=\; -2u^{1/2} + c \\
\Longrightarrow\;\; \sqrt{y} \;&=\; -\sqrt{1-x^2} + c \\
\Longrightarrow\;\; y \;&=\; \left(-\sqrt{1-x^2} + c\right)^2 .
\end{aligned}$$

Note that we cannot write $y = 1 - x^2 + c$ because $(a+c)^2 \neq a^2 + c^2$ (FOIL). We must keep the square root and c together, and square the whole thing.

To find the particular solution satisfying $y(0) = 9$, we plug in $x = 0$ and $y = 9$ into the general solution we just found and solve for the constant c. This gives

$$9 \;=\; (-1+c)^2 \quad \Longrightarrow \quad 3 \;=\; -1 + c \quad \Longrightarrow \quad c = 4.$$

Therefore, the particular solution we seek is $y = \left(-\sqrt{1-x^2} + 4\right)^2$.

6. Find the solution to $y' = (y-2)e^{\pi \sec^2(y^3) + t^4 \cos(5t)}, \;\; y(1) = 2$.

Hint: Don't separate and integrate; look for a simple function that satisfies both the ODE and initial condition.

Answer: This is a trick question and one of my favorite final exam questions. The solution is simply the equilibrium solution $y(t) = 2$ (a horizontal line). Notice that this satisfies the ODE since it gives $0 = 0$ if you plug it into both sides (derivative of a constant is 0). Since it also satisfies the initial condition, it is the solution we seek.

Note that if the initial condition had been anything other than 2 for the y-value (e.g., $y(1) = 4$), then the problem is impossible.