MATH 134 Calculus 2 with FUNdamentals

Section 9.1: Solving Differential Equations

This sections kicks off a short chapter on differential equations, a very important subject in its own right. Differential equations are used to model and understand quantities that change over time, and can be found in a wide variety of fields ranging from physics to medicine to economics to climate science.

Examples of Differential Equations

A differential equation is an equation involving an unknown function and its derivative(s). Below are four examples of some well-known differential equations:

$$\frac{dy}{dt} = ry\,, \tag{1}$$

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \tag{2}$$

$$m_i \frac{d^2\mathbf{q}_i}{dt^2} = \sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{||\mathbf{q}_i - \mathbf{q}_j||^3}, \tag{3}$$

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A + BT) - C(T - \overline{T})\,. \tag{4}$$

Equation (1) models the amount of money $y(t)$ in an account where interest is compounded continuously at an annual rate of $r\%$. It also describes a population $y(t)$ that grows exponentially with growth rate r. Equation (2) is known as the **Logistic Population Model**, where the population $P(t)$ levels off at the value K over time (K is known as the **carrying capacity**). In Equation (3) we have the Newtonian n-body problem. Here $\mathbf{q}_i$ is the position of the ith celestial body (e.g., the Sun, a planet, a comet, or a spaceship) and m_i represents the mass of the ith body. The force between each pair of bodies is given by Newton's inverse square law. This is actually a system of n differential equations, each in three dimensions, and is essentially impossible to solve without the help of a computer. The final equation models the average annual temperature $T(y,t)$ of a planet at latitude $y = \sin\theta$. In all of the above models, t represents time.

We will be focusing on **ordinary** differential equations (ODE's), which means the derivatives involved are always with respect to one quantity (usually time t). Given a differential equation, the basic aim is to find a function that **satisfies** the equation. Unlike an algebraic equation, here the goal is to find a *function*, rather than a number, that makes the equation true. For example, consider the differential equation

$$\frac{dy}{dt} = -3y\,.$$

The function $y(t) = e^{-3t}$ satisfies the ODE, as can be checked by plugging it into both sides of the equation. We have

$$\frac{dy}{dt} = -3e^{-3t} \quad \text{on the left and} \quad -3y = -3e^{-3t} \quad \text{on the right.}$$

Since these are equivalent, $y(t) = e^{-3t}$ is a solution to the ODE. Note that $y = 6e^{-3t}$ is also a solution because it too satisfies the ODE. In fact, $y = ce^{-3t}$ is a solution for any constant c because

$$\frac{dy}{dt} = c \cdot -3e^{-3t} = -3ce^{-3t} = -3y.$$

We say that the **general solution** to the differential equation is $y = ce^{-3t}$. This is a very important aspect of the subject: a differential equation has an *infinite* number of solutions (one for each value of c).

Exercise 1: Check that $y = A\sin 2t + B\cos 2t$ satisfies the ODE $y'' + 4y = 0$ for any constants A and B. Here y'' is the second derivative of y with respect to t.

Separation of Variables

We now explain a simple technique for finding the solution to a differential equation of the form

$$\frac{dy}{dt} \;=\; f(y)\cdot g(t)\,.$$

The idea is to **separate** the variables onto different sides of the equation and then **integrate** each side with respect to the given variable. Then we solve for the dependent variable (in this case y) to obtain the general solution. Here is a worked out example.

Example 1: Find the general solution to the ODE $\dfrac{dy}{dt} = -3t^2 y$ using the Separation of Variables technique (i.e., separate and integrate). Then find the particular solution satisfying the initial condition $y(0) = 5$.

Answer: We begin by moving the terms with y to the left-hand side of the equation and those with t to the right:

$$\frac{dy}{dt} \;=\; -3t^2 y \quad\Longrightarrow\quad \frac{1}{y}\,dy \;=\; -3t^2\,dt\,.$$

Next we integrate both sides, integrating on the left-hand side with respect to y and integrating on the right-hand side with respect to t. This gives

$$\int \frac{1}{y}\,dy \;=\; \int -3t^2\,dt \quad\Longrightarrow\quad \ln|y| \;=\; -t^3 + c\,.$$

Notice that we only have one integration constant c on the right-hand side. If we had a constant on the left-hand side as well (say d), we would have moved it over to the right-hand side and combined it with c (replacing $c - d$ with just c). Now we solve for y by raising both sides to the base e:

$$e^{\ln|y|} \;=\; e^{-t^3+c} \;=\; e^{-t^3}\cdot e^{c} \;=\; ce^{-t^3} \quad\Longrightarrow\quad |y| \;=\; ce^{-t^3}\,,$$

where we have replaced the constant e^c with just c (they are both arbitrary constants so we opt for the simplest choice c). Thus, $y = \pm ce^{-t^3}$, which can be condensed to just $y = ce^{-t^3}$, with $c \in \mathbb{R}$ an arbitrary constant. The general solution is $y = ce^{-t^3}$ (check that it satisfies the ODE).

To find the particular solution satisfying $y(0) = 5$, we plug in $t = 0$ and $y = 5$ into the general solution we just found and solve for the constant c. This gives

$$5 \;=\; ce^{0} \quad\Longrightarrow\quad c \;=\; 5\,.$$

Therefore, the particular solution we seek is $y = 5e^{-t^3}$.

Exercises:

2. Use the Separation of Variables technique to find the general solution to $\frac{dy}{dt} = ry$, where r is a constant. Where have we seen this formula before?

3. Use the Separation of Variables technique to find the general solution to $y' = y^2 \sin(4x)$. Then find the particular solution satisfying $y(0) = 1$.

4. Find the solution to $y' = (1 - t^2)(1 + y^2)$ satisfying the initial condition $y(0) = -1$.

5. Find the solution to $\sqrt{1 - x^2}\, y' = 2x\sqrt{y}\,, \quad y(0) = 9$.

6. Find the solution to $y' = (y - 2)e^{\pi \sec^2(y^3) + t^4 \cos(5t)}, \quad y(1) = 2$.

   Hint: Don't separate and integrate; look for a simple function that satisfies both the ODE and initial condition.