MATH 134 Calculus 2 with FUNdamentals

Section 5.9: Compound Interest and Present Value

SOLUTIONS

Exercise 1: Suppose that $P_0 = \$5{,}000$ is invested in an account paying at an annual rate of 7%. Find the amount in the account after 8 years if it is compounded (a) quarterly, (b) monthly, and (c) continuously.

Answer: (a) Use $r = 0.07, M = 4$, and $t = 8$ in the formula for compound interest:
$P(8) = 5{,}000(1 + 0.07/4)^{4\cdot 8} = \$8{,}711.07$ (rounding to the nearest cent)

(b) Use $r = 0.07, M = 12$, and $t = 8$ in the formula for compound interest:
$P(8) = 5{,}000(1 + 0.07/12)^{12\cdot 8} = \$8{,}739.13$

(c) Use $r = 0.07$ and $t = 8$ in the formula for continuously compounded interest:
$P(8) = 5{,}000e^{0.07\cdot 8} = \$8{,}753.36$

Notice the amounts **increase** in value the more often the account is compounded.

Exercise 2: How much should you invest today in order to receive \$10,000 in 5 years if interest is compounded continuously at a rate of 2.5%?

Answer: Using the formula for present value (PV), we obtain $10{,}000e^{-0.025\cdot 5} \approx \$8{,}824.97$

Exercise 3: Is it better to receive \$500 today or \$600 in 5 years if the interest rate is 3%? What if the rate increases to 4%? Assume that interest is compounded continuously.

Answer: (a) It is better to receive \$600 in 5 years if the rate is 3%. To see this, we calculate the present value of \$600 in 5 years, $600e^{-0.03*5} \approx \$516.42$. Since this amount is **greater** than \$500, it is the better deal.

(b) Somewhat surprisingly, if the interest rate bumps up to 4%, then it is better to stick with the \$500 today. This follows because the present value of \$600 in 5 years with the new interest rate is $600e^{-0.04*5} \approx \$491.24$. Since this value is **less** than \$500, it is not the better deal. Another way to see this is to compute the value of \$500 compounded continuously for 5 years at the new rate: $500e^{0.04*5} \approx \$610.70$ which is greater than \$600.

Exercise 4: Congratulations, you just won \$2 million dollars in the lottery! However, you do not get all of your money now; you will receive four yearly payments of \$500,000 beginning immediately. Assuming an interest rate of 5%, what is the present value of your prize? How much do you "lose" by not receiving the full prize today?

Answer: This is a tricky one. We need to compute the present value of each yearly payment and then add them together to see how they compare with \$2 million. The first payment starts the clock (time $t = 0$). We obtain

$$\begin{aligned} 500{,}000 + 500{,}000e^{-0.05\cdot 1} + 500{,}000e^{-0.05\cdot 2} + 500{,}000e^{-0.05\cdot 3} &= \\ 500{,}000\left(1 + e^{-0.05\cdot 1} + e^{-0.05\cdot 2} + e^{-0.05\cdot 3}\right) &\approx \$1{,}858{,}387.41 \end{aligned}$$

The "loss" on our winnings is 2 million minus the present value or a whopping \$141,612.59. You want your winnings immediately (if you can get them!)

Exercise 5: Find the PV of an income stream paying out continuously at a rate of \$750 per year for 10 years, assuming an interest rate of 5%.

Answer: We have

$$\begin{aligned} PV &= \int_0^{10} 750e^{-0.05t}\, dt \\ &= 750 \cdot \frac{1}{-0.05} e^{-0.05t} \Big|_0^{10} \\ &= -\frac{750}{0.05}\left(e^{-0.5} - 1\right) \\ &\approx \$5{,}902.04. \end{aligned}$$

Notice that this is substantially less than \$7,500.00 (\$750 for 10 years), reflecting the loss of money caused by receiving payments over time rather than immediately.

Exercise 6: Find the PV of an investment that pays out continuously at a rate of $R(t) = \$1{,}000e^{0.03t}$ per year for 8 years, assuming an interest rate of 6%.

Answer: We have

$$\begin{aligned} PV &= \int_0^{8} 1000e^{0.03t} \cdot e^{-0.06t}\, dt \\ &= \int_0^{8} 1000e^{-0.03t}\, dt \\ &= 1000 \cdot \frac{1}{-0.03} e^{-0.03t} \Big|_0^{8} \\ &= -\frac{1000}{0.03}\left(e^{-0.24} - 1\right) \\ &\approx \$7{,}112.40. \end{aligned}$$