MATH 134 Calculus 2 with FUNdamentals

Exam #1 SOLUTIONS **February 20, 2020** **Prof. Gareth Roberts**

1. Let $g(x) = \sqrt{x}$ over the interval $1 \le x \le 4$.

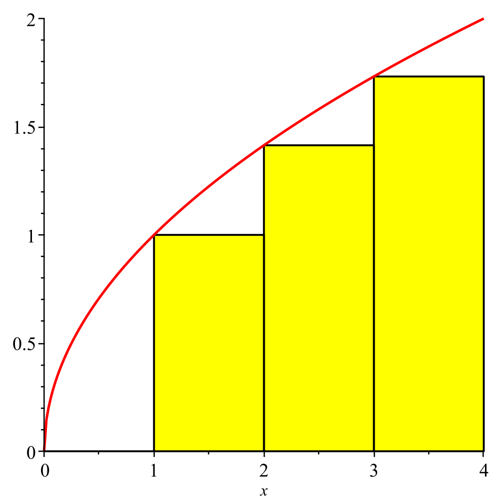
(a) Approximate the area (to three decimal places) under the graph of $g(x) = \sqrt{x}$ from $1 \le x \le 4$ by using **three** equal subintervals and left endpoints (i.e., calculate the left-hand sum L_3). (5 pts.)

Answer: The width of each rectangle is $\Delta x = (4-1)/3 = 1$. Evaluating g at the left endpoints of each subinterval gives an estimated area of

$$A \;=\; 1\,(g(1) + g(2) + g(3)) \;=\; 1 + \sqrt{2} + \sqrt{3} \;\approx\; 4.146\,.$$

(b) Sketch a graph of $g(x)$ over $[1,4]$ and draw the three rectangles used to compute L_3. Based on your figure, is your estimate in part **(a)** an underestimate, an overestimate, or can this not be determined? (5 pts.)

Answer: The value in part **(a)** is an **underestimate** because g is an increasing function (see the figure below). This can also be seen by computing the actual area (part **(d)**), since $4.146 < 14/3 = 4.\bar{6}$.



(c) Approximate the area (to three decimal places) under the graph of $g(x) = \sqrt{x}$ from $1 \le x \le 4$ by using **three** equal subintervals and midpoints (i.e., calculate the midpoint sum M_3). (5 pts.)

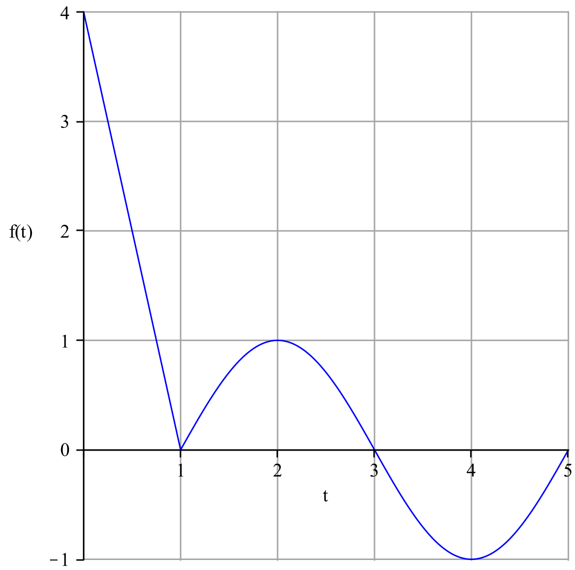
Answer: The width of each rectangle is still 1. Evaluating g at the midpoints of each subinterval gives an estimated area of

$$A \;=\; 1\,(g(1.5) + g(2.5) + g(3.5)) \;=\; \left(\sqrt{1.5} + \sqrt{2.5} + \sqrt{3.5}\right) \;\approx\; 4.677\,.$$

(d) Use a definite integral to compute the **exact** area under the graph of $g(x) = \sqrt{x}$ from $1 \le x \le 4$. (5 pts.)

Answer: $A = \displaystyle\int_1^4 \sqrt{x}\,dx = \int_1^4 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_1^4 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}(8-1) = \frac{14}{3}\,.$

2. Define $A(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 5$, where the graph of $f(t)$ is shown below.



(a) Find $A(0)$ and $A(1)$. (4 pts.)

Answer: The function $A(x)$ represents the area under the curve from 0 to x. Thus $A(0) = 0$ and $A(1) = \frac{1}{2} \cdot 1 \cdot 4 = 2$.

(b) Find each of the following quantities, if they exist. (8 pts.)

$A'(1) =$ <u>0</u> $\qquad A''(1) =$ <u>D.N.E.</u>

$A'(2) =$ <u>1</u> $\qquad A''(2) =$ <u>0</u>

Answer: Using FTC, part 2, we have that $A'(x) = f(x)$ and thus $A'(1) = f(1) = 0$ and $A'(2) = f(2) = 1$, as can be seen from the graph. Then, $A'(x) = f(x)$ implies that $A''(x) = f'(x)$ (differentiate both sides with respect to x). This means that $A''(1) = f'(1)$ does not exist (corner at $t = 1$) and $A''(2) = f'(2) = 0$ (horizontal tangent line at $t = 2$.)

(c) On what interval(s) is $A(x)$ increasing? (3 pts.)

Answer: Since $A'(x) = f(x)$ and since a function is increasing whenever its derivative is positive, we see that A is increasing whenever $f > 0$ (graph above the axis), or when $0 < x < 1$ or $1 < x < 3$.

(d) On what interval(s) is $A(x)$ concave down? (3 pts.)

Answer: Since $A''(x) = f'(x)$ and since a function is concave down whenever its second derivative is negative, we see that A is concave down whenever $f' < 0$ (negative slope), or when $0 < x < 1$ or $2 < x < 4$.

3. Evaluate each of the following integrals. Note that part **(d)** is a **definite integral**. (22 pts.)

(a) $\displaystyle\int e^{3x} + \sec^2 x - \frac{3}{x^2}\, dx$

Answer: Computing each antiderivative separately using the appropriate formula, we obtain

$$\frac{1}{3}e^{3x} + \tan x + 3x^{-1} + c,$$

where the last term follows from the power rule applied to $-3x^{-2}$.

(b) $\displaystyle\int \frac{2t+1}{4t^2+4t+3}\, dt$

Answer: This is a u-substitution with $u = 4t^2+4t+3$. Then $du = 8t+4dt = 4(2t+1)\,dt$. This gives $du/4 = 2t+1\,dt$, which transforms the integral into

$$\int \frac{1}{u} \cdot \frac{du}{4} = \frac{1}{4}\ln|u| + c\,.$$

Converting back into the original variable gives

$$\frac{1}{4}\ln|4t^2+4t+3| + c\,.$$

(c) $\displaystyle\int xe^{-x^2}\, dx$

Answer: Letting $u = -x^2$, we have $du = -2x\,dx$ or $\frac{du}{-2} = x\,dx$. The integral becomes

$$\int e^u \cdot \frac{du}{-2} = -\frac{1}{2}e^u + c\,.$$

Converting back into the original variable gives $-\dfrac{1}{2}e^{-x^2} + c\,.$

(d) $\displaystyle\int_0^{\pi/2} \cos^4 x \sin x\, dx$

Answer: This is a u-substitution with $u = \cos x$. Then $du = -\sin x\,dx$ or $-du = \sin x\,dx$. Also, if $x = 0$, then $u = \cos(0) = 1$ and if $x = \pi/2$, then $u = \cos(\pi/2) = 0$. In the u-variable, the integral becomes

$$-\int_1^0 u^4\,du = \int_0^1 u^4\,du = \frac{u^5}{5}\bigg|_0^1 = \frac{1}{5}\,.$$

4. Suppose that the acceleration of a particle traveling along a line is given by

$$a(t) = 3\sin(3t) + 8t.$$

If the initial velocity is $v(0) = 5$ and the initial position is $s(0) = -2$, find the position function $s(t)$. (10 pts.)

Answer:

To find $v(t)$ we compute the antiderivative of the acceleration. This is trickier than it looks. We need to use the fact that

$$\int \sin(kt)\, dt \;=\; -\frac{1}{k}\cos(kt) + c\,,$$

which is true for any constant k (check it with the chain rule.) Thus, we have that

$$v(t) \;=\; 3 \cdot -\frac{1}{3}\cos(3t) + 4t^2 + c \;=\; -\cos(3t) + 4t^2 + c\,.$$

Since $v(0) = 5$, we find that $5 = -\cos(0) + 0 + c$ or $5 = -1 + c$, which implies that $c = 6$. Thus,

$$v(t) \;=\; -\cos(3t) + 4t^2 + 6\,.$$

Next, we compute another antiderivative to find the position function $s(t)$. This time we use the formula

$$\int \cos(kt)\, dt \;=\; \frac{1}{k}\sin(kt) + c$$

and obtain

$$s(t) \;=\; -\frac{1}{3}\sin(3t) + \frac{4}{3}t^3 + 6t + c\,.$$

Finally, using the initial position $s(0) = -2$, we have that $-2 = -(1/3)\sin(0) + 0 + 0 + c$, which implies that $c = -2$. The final answer is

$$s(t) \;=\; -\frac{1}{3}\sin(3t) + \frac{4}{3}t^3 + 6t - 2\,.$$

5. Evaluate $\displaystyle\int_0^{2/3} \frac{3}{9x^2+4}\, dx$ using the substitution $u = \dfrac{3}{2}x$. Give the **exact** answer (no decimals). (10 pts.)

Answer: Letting $u = \frac{3}{2}x$, we have $x = \frac{2}{3}u$ and $dx = \frac{2}{3}\, du$. Then,

$$9x^2 + 4 \;=\; 9\left(\frac{2}{3}u\right)^2 + 4 \;=\; 9 \cdot \frac{4}{9}u^2 + 4 \;=\; 4u^2 + 4 \;=\; 4(u^2+1).$$

Also, if $x = 0$, then $u = 0$, and if $x = 2/3$, then $u = \frac{3}{2} \cdot \frac{2}{3} = 1$.

Applying the above calculations, the integral transforms to

$$\int_0^1 \frac{3}{4(u^2+1)} \cdot \frac{2}{3}\, du \;=\; \frac{1}{2}\int_0^1 \frac{1}{u^2+1}\, du \;=\; \frac{1}{2}\tan^{-1} u\,\Big|_0^1 \;=\; \frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1}(0)\right) \;=\; \frac{\pi}{8}\,,$$

since $\tan^{-1}(1) = \pi/4$ and $\tan^{-1}(0) = 0$.

6. **Calculus Potpourri:** (20 pts.)

(a) Suppose that $\int_0^4 f(x)\,dx = -3$ and $\int_0^7 f(x)\,dx = 6$, and that $f(x)$ is an **even** continuous function. Find the value of $\int_{-7}^{-4} \pi f(x)\,dx$.

Answer: First, using linearity, we have

$$\int_0^7 f(x)\,dx \;=\; \int_0^4 f(x)\,dx + \int_4^7 f(x)\,dx\,,$$

which implies

$$6 \;=\; -3 + \int_4^7 f(x)\,dx \quad \text{and thus} \quad \int_4^7 f(x)\,dx \;=\; 9\,.$$

Since f is an even function, it is symmetric with respect to the y-axis. This means the integral of f over an interval on one side of the y-axis is equivalent to the integral of f over the reflection of that interval onto the other side of the axis. Thus, we have

$$\int_{-7}^{-4} f(x)\,dx \;=\; 9 \quad\Longrightarrow\quad \int_{-7}^{-4} \pi f(x)\,dx \;=\; 9\pi\,,$$

since constants pull out of integrals.

(b) A particle travels in a straight line with velocity $v(t) = 4 - t^2$ m/s. Find the **total distance traveled** by the particle over the time interval $[0, 3]$.

Answer: To find the total distance traveled, we compute $\int_0^3 |v(t)|\,dt = \int_0^3 |4 - t^2|\,dt$. In order to evaluate this integral, we need to determine where $v(t)$ is positive and where it is negative. To do this we solve $v(t) = 0$ or $4 - t^2 = 0$. This gives $t^2 = 4$ or $t = \pm 2$. Since the time interval is $[0, 3]$, we disregard $t = -2$. The graph of $4 - t^2$ is a parabola opening down with vertex at $(0, 4)$, so v is positive on $0 \le t < 2$ and negative for $2 < t \le 3$. Therefore,

$$\begin{aligned}
\int_0^3 |4 - t^2|\,dt &= \int_0^2 4 - t^2\,dt \;-\; \int_2^3 4 - t^2\,dt \\
&= 4t - \frac{t^3}{3}\Big|_0^2 - \left(4t - \frac{t^3}{3}\Big|_2^3\right) \\
&= 8 - \frac{8}{3} - \left(12 - 9 - \left(8 - \frac{8}{3}\right)\right) \\
&= \frac{16}{3} - \left(3 - \frac{16}{3}\right) \\
&= \frac{32}{3} - 3 \\
&= \frac{23}{3}\ \text{m.}
\end{aligned}$$

(c) Find and simplify $\frac{d}{dx}\left(\int_{x^3}^{1} \ln(t^3 + 2020)\,dt\right)$.

Answer: This is a problem using FTC, part 2 and the chain rule. First flip the limits of integration and then apply FTC, part 2 as well as the chain rule. The solution is

$$
\begin{aligned}
\frac{d}{dx}\left(\int_{x^3}^{1} \ln(t^3 + 2020)\,dt\right) &= -\frac{d}{dx}\left(\int_{1}^{x^3} \ln(t^3 + 2020)\,dt\right) \\
&= -\ln((x^3)^3 + 2020) \cdot \frac{d}{dx}(x^3) \\
&= -3x^2 \ln(x^9 + 2020)\,.
\end{aligned}
$$