Key

MATH 135-08, 135-09 Calculus 1, Fall 2017

Understanding Functions: Worksheet for Section 1.1

The first chapter of the course text reviews the key material from precalculus that will be crucial to understanding calculus. The main points of the first section are described in this handout. Please read the handout carefully and complete the exercises.

1.1 Real Numbers, Functions, and Graphs

A **function** is a rule that assigns to each input element in the **domain** a unique output element in the **range**. The set of inputs to a function is called the **domain**, while the set of outputs is called the **range**. If a function $f : A \mapsto B$ maps from the set A to the set B (but not necessarily all of B), then we often call B the **co-domain**.

We will use four different methods to describe a function: analytically (an explicit formula), graphically, numerically (table), and verbally (described in words). Typically, we will use x and t as the independent variables (inputs) and letters such as y, N, s (for position), v (for velocity) or a (for acceleration) as the dependent variables (outputs). When graphing a function, we will always assume the independent variable is plotted on the horizontal axis while the dependent variable is plotted on the vertical axis. In order to represent a function, a graph must pass the **vertical line test**, that is, any vertical line through the graph can only pass through at most one point. Otherwise, one input in the domain would have more than one output in the range, violating the definition of a function.

Exercise 0.1 *Find the domain and range of each of the following functions:*

(a) $f(x) = 7x - 3$

D: $\mathbb{R}$ or $(-\infty, \infty)$

R: $\mathbb{R}$ or $(-\infty, \infty)$

(b) $g(\theta) = \sin\theta$

D: $\mathbb{R}$

R: $[-1, 1]$

(c) $h(t) = \sqrt{t^2 - 1} - 3$

D: Solve $t^2 - 1 \geq 0$

$(t-1)(t+1) \geq 0$

$+ \quad -1 \quad - \quad 1 \quad +$

$\{t : t \geq 1 \text{ or } t \leq -1\}$

or $(-\infty, -1] \cup [1, \infty)$

R: $[-3, \infty)$

We say that a function f is **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

Likewise, f is **decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

This is much easier to see visually. Increasing functions move upwards from left to right while decreasing functions move downwards from left to right.

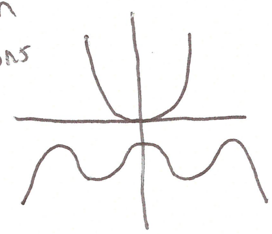
A function that satisfies $f(-x) = f(x)$ for all x in its domain is called an **even** function. The graph of an even function is symmetric about the vertical axis. If a function satisfies $f(-x) = -f(x)$ for all x in its domain, then it is an **odd** function. The graph of an odd function is symmetric about the origin (after reflecting about both the horizontal and vertical axes, the same graph is obtained.)

Examples of even functions: $7,\ x^2,\ x^4,\ |x|,\ x^{-2},\ \cos x$.

Examples of odd functions: $x,\ x^3,\ x^5,\ 1/x,\ \sin x,\ \tan x$.

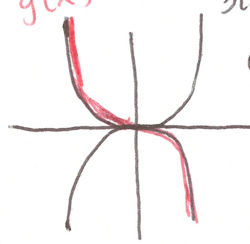
Exercise 0.2 *Below, sketch the graph of an even function, an odd function, and a function that is neither even nor odd.*

Even functions

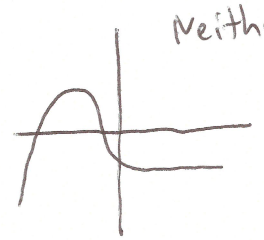


$g(x) = -x^3$ $f(x) = x^3$

Odd



Neither



Exercise 0.3 (Challenge Problem) *Can a function be* **both** *even and odd?* *__Hint:__ What would it look like?*

Yes! $f(x) = 0$ satisfies both $f(x) = f(-x)$ and $f(-x) = -f(x)$

The Absolute Value Function $f(x) = |x|$

One of the most important functions in calculus (and in much of mathematics) is the **absolute value** function. The graph of this function is a V with vertex at the origin. Although you may have learned that the absolute value is always positive, this hardly captures the meaning of this function. **The absolute value is used to measure distance.** For example, $|4| = 4$ and $|-4| = 4$ both indicate that the points 4 and -4 are each four units from 0 on the number line. In general, the expression $|a - b|$ gives the distance between the numbers a and b on the number line. Thus, $|2 - 5| = 3$ since 2 and 5 are 3 units apart on the number line. Similarly, $|3 + 4| = |3 - -4| = 7$ since 3 and -4 are 7 units apart.

The piecewise definition for $|x|$ is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

You should memorize this definition. It simply says that if x is positive, then $|x|$ is itself, x. But if x is negative, then $|x|$ is the opposite of itself, $-x$.

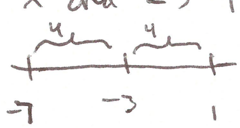
The key to understanding expressions with absolute values is to think in terms of distance. For example, $|x - 5| < 3$ means that the distance between the numbers x and 5 is less than 3. In other words, the solution to this inequality is the set of x-values that are less than 3 units away from 5. If you draw a number line, you can see that these are all the numbers between 2 and 8, that is $2 < x < 8$ or, using interval notation, the solution is $(2, 8)$.

One important observation about absolute values is that $|x| < r$ is equivalent to the inequality $-r < x < r$, or the interval $(-r, r)$. Again, this is easy to understand in terms of distance because $|x - 0| = |x|$, so $|x| < r$ just means the set of points that are less than r away from the origin.

Exercise 0.4 *Translate the following expression into words and then solve the inequality.*

(a) $|x + 3| < 4$

The distance between x and -3 is less than 4.



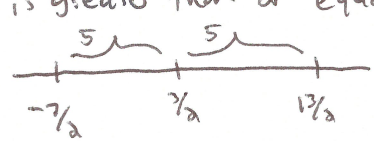
$-7 < x < 1$

or

$(-7, 1)$

(b) $|x - 3/2| \geq 5$.

The distance between $3/2$ and x is greater than or equal to 5.



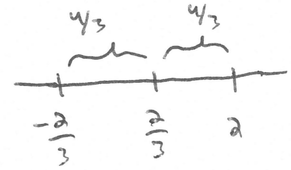
$\{x : x \geq 13/2 \text{ or } x \leq -7/2\}$

$[13/2, \infty) \cup (-\infty, -7/2]$

Exercise 0.5 *Find all x that satisfy the inequality $|3x-2| \leq 4$. Express your answer in interval notation.*

use $|a \cdot b| = |a| \cdot |b|$. $\quad |3x-2| = |3(x-2/3)| = 3|x-2/3|$

$3|x-2/3| \leq 4$ or $|x-2/3| \leq 4/3$

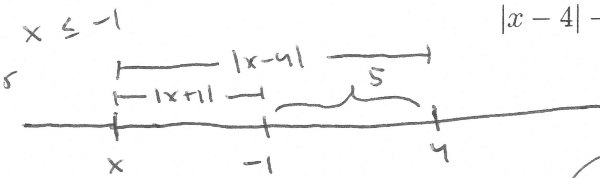


$[-2/3, 2]$

Exercise 0.6 (Challenge Problem) *Find all x-values that satisfy the following equation.*
Hint: Interpret the equation in terms of distance and draw a picture.

$$|x-4| - |x+1| = 5$$

Any $x \leq -1$ works



Distance b/w x and 4
minus " " " " −1
must equal 5.

$x \leq -1$ or $(-\infty, -1]$

Shifting and Scaling functions

One key idea in mathematics is to apply a transformation to a graph (or function) and convert it into a new, but related graph. The simplest way to do this is to shift the graph (translation) or scale it (stretch or compress).

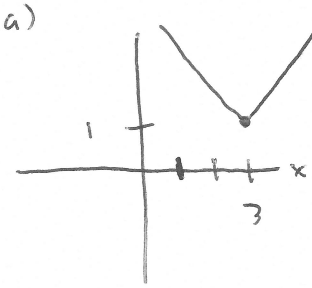
Here are the basic rules for shifting a graph vertically or horizontally. The constant c is assumed to be a positive real number.

- $f(x) + c$ shifts the graph of $f(x)$ upward by c units.
- $f(x) - c$ shifts the graph of $f(x)$ downward by c units.
- $f(x + c)$ shifts the graph of $f(x)$ to the left by c units.
- $f(x - c)$ shifts the graph of $f(x)$ to the right by c units.

Notice the difference between the vertical and horizontal shifts. The vertical shifts occur by adding/subtracting c **outside** the parentheses. This effects the output of the function—a range change. Meanwhile, the horizontal shifts occur by adding/subtracting c **inside** the parentheses. This alters what goes into the function—a domain change. For a domain change, it is the x-axis that gets shifted. For example, for the function $f(x + 2)$, we are really shifting the x-axis two units to the right, which has the effect of shifting the graph of f two units to the left.

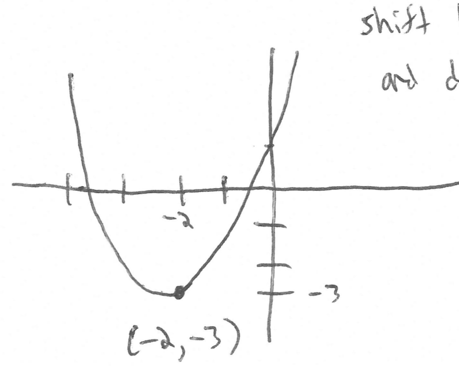
Exercise 0.7 **(a)** *Starting with the graph of $f(x) = |x|$, sketch the graph of $g(x) = |x-3| + 1$.*
(b) *Starting with the graph of $h(x) = x^2$, sketch the graph of $j(x) = (x+2)^2 - 3$.*

(a)



shift right 3 units
and up 1 unit

(b)



shift left 2 units
and down 3 "

Here are the basic rules for scaling a graph vertically (range change) or horizontally (domain change). Below we assume that the constant c satisfies $c > 1$.

- $cf(x)$ stretches the graph of $f(x)$ vertically by a factor of c units.
- $\frac{1}{c} \cdot f(x)$ compresses the graph of $f(x)$ vertically by a factor of c units.
- $f(cx)$ compresses the graph of $f(x)$ horizontally by a factor of c units.
- $f(\frac{1}{c} \cdot x)$ stretches the graph of $f(x)$ horizontally by a factor of c units.

Reflections

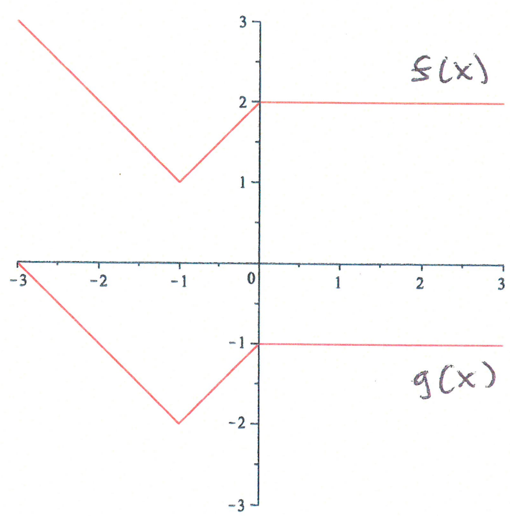
The transformation $-f(x)$ reflects the graph of $f(x)$ about the x-axis. Each point (x, y) on the graph of $y = f(x)$ is mapped to the point $(x, -y)$ on the graph of $y = -f(x)$.

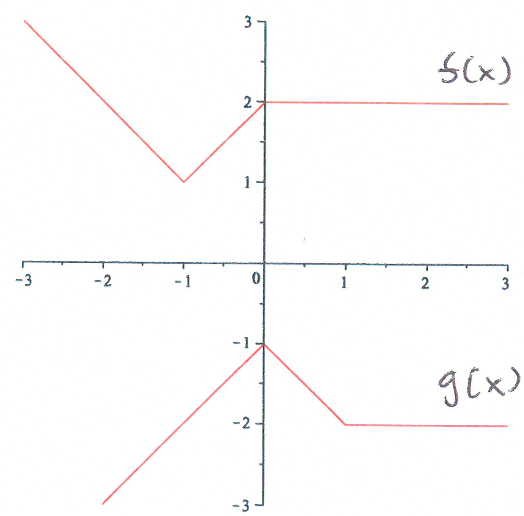
The transformation $f(-x)$ reflects the graph of $f(x)$ about the y-axis. Each point (x, y) on the graph of $y = f(x)$ is mapped to the point $(-x, y)$ on the graph of $y = f(-x)$.

Note that since $f(x) = f(-x)$ for an even function, the graph of an even function is symmetric with respect to the y-axis. Simply put, the equation $f(x) = f(-x)$ means that reflecting the graph of $f(x)$ about the y-axis has no effect on the graph.

Similarly, an odd function is defined by $f(-x) = -f(x)$, which is equivalent to $f(x) = -f(-x)$. This means that the graph of an odd function is symmetric with respect to the origin. In other words, reflecting the graph of $f(x)$ about the x-axis and then again about the y-axis has no effect on the graph.

Exercise 0.8 *For each of the graphs shown below, one or more transformations have been applied to the original function $f(x)$ (top graph) to obtain a new function $g(x)$ (bottom graph). In mathematical terms, state the formula for $g(x)$ in terms of $f(x)$. For example, a typical answer might be $g(x) = 4f(x + 7)$.*





(a) $g(x) =$ $f(x) - 3$

Shift down 3 units

(b) $g(x) =$ $-f(x-1)$

Shift right one unit and reflect about x-axis