MATH 135-08, 135-09 Calculus 1, Fall 2017

Evaluating Limits Algebraically: Worksheet for Section 2.5

If a function is continuous at a point, then the limit of the function is easily found by simple substitution. For instance, $\lim_{x\to 4} x^2 - 3 = 4^2 - 3 = 13$. Otherwise, we may use a graph to try and discern the limit or perform numerical calculations (plug in values really, really close to the point $x = a$) to find the limit of the function as x approaches a.

This section is concerned with a new technique, namely, using algebra to calculate limits. Here is a simple example. Consider the problem

$$\lim_{x\to 3} \frac{x-3}{x^2-9} .$$

The function is clearly not continuous at $x = 3$ because the denominator becomes 0 when $x = 3$. However, notice that the numerator is also 0 when $x = 3$. Thus, our limit has the form $\frac{0}{0}$, which is called an **indeterminate form**. In this case, there is often some algebra (e.g., factoring) that can be performed to simplify the function and compute the limit by hand.

We have

$$\lim_{x\to 3} \frac{x-3}{x^2-9} = \lim_{x\to 3} \frac{x-3}{(x-3)\cdot(x+3)} = \lim_{x\to 3} \frac{1}{x+3} = \frac{1}{6}.$$

The cancellation is valid here because we never reach $x = 3$ in the limit, so that $x - 3 \neq 0$ and can be cancelled from the top and bottom of the fraction.

Here is a list of indeterminate forms. A limit that takes one of these forms can be *anything* (hence the name indeterminate), so further work must be done to find the actual value of the limit.

$$\textbf{Key indeterminate forms: } \frac{0}{0},\ \frac{\infty}{\infty},\ \infty\cdot 0,\ \infty-\infty. \tag{1}$$

Exercise 0.1 *Find the value of the limit by first canceling a common factor from the numerator and denominator.*

$$\lim_{x\to 2} \frac{x^2-4}{x^2+4x-12}$$

Exercise 0.2 *If* $f(x) = 5x^2 - 3x$, *find the value of* $\lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$. *Limits of this form are very important in Calculus.*

Exercise 0.3 *Find the value of* $\lim_{x\to 9} \frac{\sqrt{x}-3}{9-x}$. **Hint:** *Multiply top and bottom by the conjugate* $\sqrt{x}+3$.

Exercise 0.4 *Find the value of* $\lim_{\theta\to\pi/2} \frac{\tan\theta}{\sec\theta}$.

Exercise 0.5 *Find the value of each one-sided limit.* **Hint:** *Use the definition of the absolute value function and the property* $|ab| = |a||b|$.

(i) $\lim_{x\to 3^-} \frac{|4x-12|}{x-3}$

(ii) $\lim_{x\to 3^+} \frac{|4x-12|}{x-3}$

Exercise 0.6 *Find the value of* $\lim_{t\to 1} \frac{6}{t^2-1} - \frac{3}{t-1}$. **Hint:** *Add the fractions and simplify.*