MATH 135-08, 135-09 Calculus 1, Fall 2017

Product and Quotient Rules: Worksheet for Section 3.3

There are two useful rules for computing the derivative of a product and quotient of two functions. Interestingly, Leibniz himself messed up the product rule in an early draft of his manuscript on calculus.

Theorem 0.1 (Product Rule) *If $f(x)$ and $g(x)$ are differentiable functions, then so is their product $f(x) \cdot g(x)$. The derivative of the product is given by*

$$\boxed{\frac{d}{dx}\left(f(x) \cdot g(x)\right) \;=\; f'(x)g(x) + f(x)g'(x)\,.} \tag{1}$$

The shorthand notation for the product rule is $(fg)' = f'g + fg'$. Notice the symmetry in Formula (1) and that it is **not** the case that the derivative of the product equals the product of the derivatives. For instance, suppose we applied the Product Rule to take the derivative of $x \cdot x$. If we just multiplied the product of the derivatives, we would get

$$\frac{d}{dx}\left(x \cdot x\right) \;=\; \frac{d}{dx}\left(x\right) \cdot \frac{d}{dx}\left(x\right) \;=\; 1 \cdot 1 \;=\; 1 \;???$$

But this is clearly incorrect since $x \cdot x = x^2$ and the derivative of x^2 is $2x$ by the Power Rule. A correct application of the Product Rule is as follows:

$$\frac{d}{dx}\left(x \cdot x\right) \;=\; \frac{d}{dx}\left(x\right) \cdot x + x \cdot \frac{d}{dx}\left(x\right) \;=\; 1 \cdot x + x \cdot 1 \;=\; 2x.$$

Exercise 0.2 *Use the Product Rule to find $f'(x)$ where $f(x) = (3x^2 + 1)e^x$. Simplify your answer.*

Theorem 0.3 (Quotient Rule) *If $f(x)$ and $g(x)$ are differentiable functions, then so is their quotient $f(x)/g(x)$ as long as $g(x) \neq 0$. The derivative of the quotient is given by*

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \;=\; \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\,.} \tag{2}$$

Note: The Quotient Rule can be derived from the Product Rule. Start by letting $Q(x) = \dfrac{f(x)}{g(x)}$. Then cross multiply and differentiate both sides with respect to x using the Product Rule. Solving for $Q'(x)$ leads to formula (2).

Exercise 0.4 *Use the Quotient Rule to calculate the derivative of* $\frac{1}{x^4}$ *and check your answer against the result obtained from the Power Rule.*

Exercise 0.5 *If* $g(x) = \frac{3x+1}{2x-5}$, *find and simplify* $g'(x)$.

Exercise 0.6 *If* $h(x) = \frac{e^x}{x^2+1}$, *find and simplify* $h'(x)$.

Exercise 0.7 *Suppose that* $f(3) = 5, f'(3) = -7, g(3) = 2$ *and* $g'(3) = 1/2$. *If* $H(x) = \frac{f(x)}{xg(x)}$, *find* $H'(3)$.