Key

MATH 135-08, 135-09 Calculus 1, Fall 2017

Graph Sketching and Asymptotes: Worksheet for Section 4.6

Key Idea: The point of this section is to combine all of the information obtained from the first and second derivatives (intervals where the function is increasing, decreasing, concave up or down, critical points, inflection points) and use it to draw a graph of the function. Horizontal and vertical asymptotes are also important to locate when sketching a graph.

Example 1: Consider the function $f(x) = x^6 - 6x^4$. Find and classify all critical points. Find any inflection points. Use the first and second derivatives to sketch the graph of f.

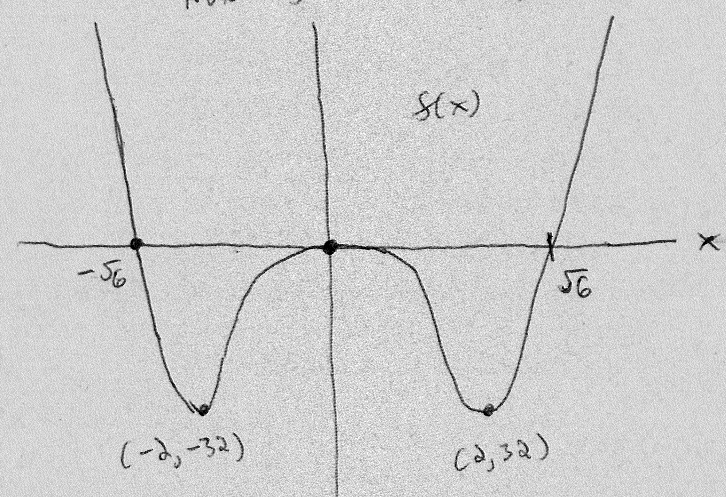
$f'(x) = 6x^5 - 24x^3 = 6x^3(x^2-4) \Rightarrow x = 0, 2, -2$ are critical points

$f''(x) = 30x^4 - 72x^2 = 6x^2(5x^2-12) \Rightarrow x = 0, \pm\sqrt{\frac{12}{5}}$ are possible inflection pts.

Note: f even $\Rightarrow$ symmetric about y axis

f': decr. $-$ | -2 | incr. $+$ | 0 | decr. $-$ | 2 | incr. $+$

f'': up $+$ | $-\sqrt{\frac{12}{5}}$ | down $-$ | 0 | down $-$ | $\sqrt{\frac{12}{5}} \approx 1.55$ | up $+$



$\therefore$ $x = -2, 2$ local mins

$x = 0$ local max

$x = \pm\sqrt{\frac{12}{5}}$ inflection pts.

Functions with Asymptotes:

Recall that $f(x)$ has a **vertical asymptote** at $x = b$ if either $\lim_{x\to b^+} f(x) = \pm\infty$ or $\lim_{x\to b^-} f(x) = \pm\infty$. These limits need to be in agreement with the information obtained from the first and second derivatives. A function $f(x)$ has a **horizontal asymptote** at $y = k$ if $\lim_{x\to\infty} f(x) = k$ or $\lim_{x\to-\infty} f(x) = k$. In the first case, the asymptote occurs to the far right of the graph, while in the second case, the asymptote appears to the far left.

Example 2: Identify the vertical and horizontal asymptotes of the function $F(x) = \dfrac{3x^2 + 5x - 7}{2x^2 - 10}$.

vertical asymptotes occur where the denominator $= 0$
(assuming the numerator is $\neq 0$ there)

$2x^2 - 10 = 0 \Rightarrow x^2 = 5$

$x = \pm\sqrt{5}$

Horiz. asymp: $y = 3/2$

$\lim_{x\to\infty} \dfrac{3x^2+5x-7}{2x^2-10} = \dfrac{3}{2}$

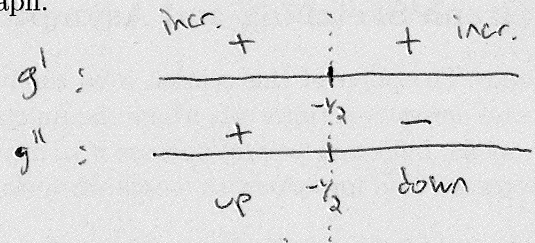
using L'Hôpital's rule or $\div$ by x^2 (Ch. 2 techniques)

Example 3: Identify the vertical and horizontal asymptotes of $g(x) = \dfrac{5x-3}{2x+1}$ and then use the first and second derivatives of g to sketch its graph.

Asymptotes:

vertical: $x = -1/2$

Horiz.: $y = 5/2$



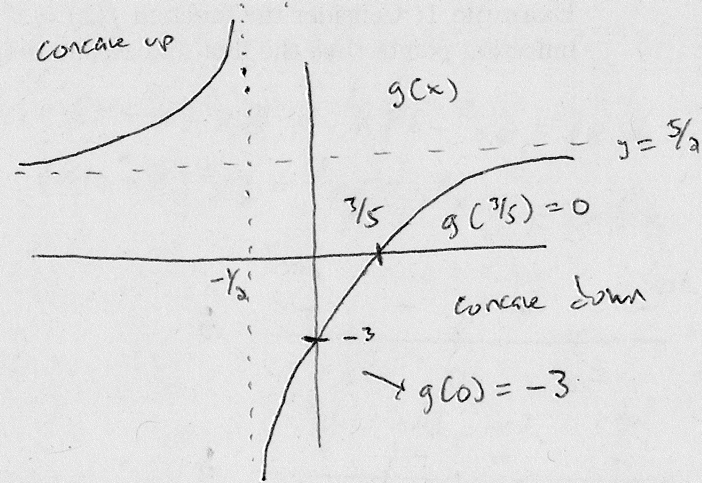
$$g'(x) = \frac{(2x+1)5 - (5x-3)\cdot 2}{(2x+1)^2}$$

$$= \frac{10x + 5 - 10x + 6}{(2x+1)^2}$$

$$= \frac{11}{(2x+1)^2} > 0 \Rightarrow g \text{ always increasing}$$

$$g''(x) = -22(2x+1)^{-3}\cdot 2 = \frac{-44}{(2x+1)^3}$$

chain rule



Example 4: Carefully find and simplify the first and second derivatives of $f(x) = xe^{-x}$. Use this information to find the critical and inflection points of f. Sketch the graph of f. Are there any asymptotes? *Hint:* Use L'Hôpital's Rule.

Note: $e^{x} > 0$ for all x

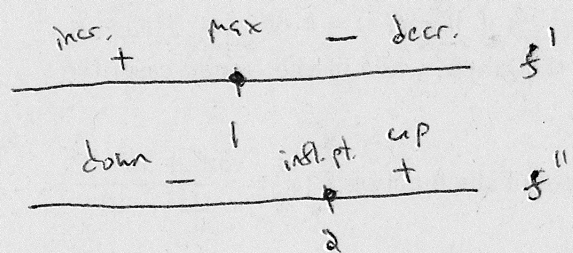
so $e^{-x} > 0$

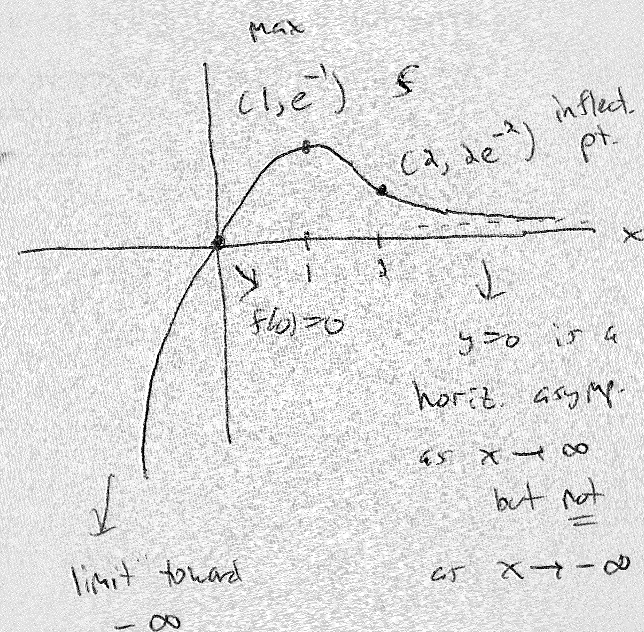
$$f'(x) = e^{-x} + x\cdot e^{-x}\cdot(-1) = e^{-x}(1-x)$$

$$f''(x) = -e^{-x}(1-x) + e^{-x}(-1) = e^{-x}(x-2)$$

$\Rightarrow$ $x=1$ is a crit. pt and

$x = 2$ is possible infl. pt.





Also, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{x}{e^x}$

$= \lim_{x\to\infty} \dfrac{1}{e^x}$ by L'Hôpital

$= 0$

$\underline{\text{but}}$ $\lim_{x\to -\infty} f(x) = -\infty$