MATH 135-08, 135-09 Calculus 1, Fall 2017

The Second Derivative and Its Implications

Worksheet for Section 4.4

Key Idea: This section focuses on using the second derivative to understand properties of a function such as where it is concave up or concave down. The second derivative can also be used to determine whether a critical point is a local maximum or minimum. Understanding both the first and second derivative of a function enables us to draw a very detailed graph of a function.

Recall that the second derivative $f''(x)$ indicates whether a function is concave up ($f'' > 0$) or concave down ($f'' < 0$). Concave up means that the first derivative is increasing (slopes increasing), while concave down means that the first derivative is decreasing (slopes decreasing).

$$f''(x) > 0 \text{ for } x \in (a,b) \implies f \text{ is concave up on } (a,b)$$
$$f''(x) < 0 \text{ for } x \in (a,b) \implies f \text{ is concave down on } (a,b)$$

An **inflection point** is a point where the concavity changes. It can be found by solving the equation $f''(x) = 0$ and then checking that the sign of f'' flips to either side of a solution.

Note: It is helpful to draw a second derivative number line and indicate where f'' is positive, negative, or 0.

Example 1: The function $f(x) = (x-2)^3$ has one inflection point. Find it. On what interval is f concave up? concave down? Sketch the graph of f.
Hint: This functions is a shifted version of another function you should already know how to graph.

The second derivative can also be used to determine the type of critical point when f'' exists. For example, if $x = c$ is a critical point (so $f'(c) = 0$), and $f''(c) > 0$, then $x = c$ is a local minimum because the function is concave up at $x = c$. (The sign of the derivative goes from $-$, to 0 at $x = c$, then becomes $+$ for $x > c$, so by the first derivative test, c is a local min.)

Second Derivative Test: Suppose that $x = c$ is a critical point of f.

$f''(c) > 0$	$\implies$	c is a local min
$f''(c) < 0$	$\implies$	c is a local max
$f''(c) = 0$	$\implies$	test is inconclusive

Note: If c is a critical point and $f''(c) = 0$, then c may either be a local max, a local min, or neither. Here is an example with three functions having a critical point with zero second derivative; yet the critical point is a different type in each case.

Example 2: Consider the three functions $f(x) = x^4$, $g(x) = -x^4$, and $h(x) = x^3$. Show that $x = 0$ is a critical point for each function and that $f''(0) = g''(0) = h''(0) = 0$. Thus, the second derivative test is inconclusive in all three cases. Graph each function (without a calculator) to determine the type of critical point (local max, local min, or neither) at $x = 0$ for each function.

Example 3: For the function $F(x) = \dfrac{1}{x^2 - x + 2}$, find and simplify $F'(x)$ and $F''(x)$. Then find the critical point(s) of F and use the second derivative test to determine whether each critical point is a local max, local min, or neither.

Example 4: Let $g(x) = x^4 - 4x^3$. Find and classify all critical points. Find any inflection points and give the intervals on which the function is concave up or down. Use the first and second derivatives to sketch the graph of g.

Example 5: Use the first and second derivatives to sketch the graph of $y = \sin x + \frac{1}{2}x$ over the interval $[0, 2\pi]$. Identify all critical points and inflection points on your graph.