# MATH 136-03 Calculus 2, Spring 2019

## Section 10.6: Power Series

## Solutions

**Exercises:** Find the radius $R$ and interval of convergence for each of the following power series. Be sure to check the endpoints.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

   **Answer:** The center of the series is $c = 0$. Let $a_n = \frac{x^n}{2^n}$. We apply the ratio test regarding $x$ as some fixed value. We find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}}\right| = \left|\frac{x^{n+1}}{2^{n+1}}\right| \cdot \left|\frac{2^n}{x^n}\right| = \frac{|x^n \cdot x|}{2^n \cdot 2} \cdot \frac{2^n}{|x^n|} = \frac{|x|}{2}.$$

   Then, since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{2} = \frac{|x|}{2}$, we solve $|x|/2 < 1$ to find where the power series converges (by the ratio test). This yields $|x| < 2$ and thus the radius of convergence is $R = 2$. The power series converges for $-2 < x < 2$ and diverges for $|x| > 2$. However, we must check the endpoints $x = 2$ and $x = -2$ directly to determine if the series converges at these points.

   Substituting $x = -2$ into the original series gives

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n,$$

   which diverges by the $n$th term test. Substituting $x = 2$ into the original series gives

$$\sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1,$$

   which also diverges by the $n$th term test. Therefore, the interval of convergence for the power series is $-2 < x < 2$ or $(-2, 2)$.

2. $\displaystyle\sum_{n=1}^{\infty} (x-1)^n \frac{1}{n\,3^n}$

   **Answer:** The center of the series is $c = 1$. Let $a_n = \frac{(x-1)^n}{n3^n}$. We apply the ratio test regarding $x$ as some fixed value. We find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-1)^{n+1}}{(n+1)3^{n+1}}}{\frac{(x-1)^n}{n3^n}}\right| = \left|\frac{(x-1)^{n+1}}{(n+1)3^{n+1}}\right| \cdot \left|\frac{n3^n}{(x-1)^n}\right| = \frac{|(x-1)^n \cdot (x-1)|}{(n+1)3^n \cdot 3} \cdot \frac{n3^n}{|(x-1)^n|} = \frac{|x-1|}{3} \cdot \frac{n}{n+1}.$$

   Then, since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-1|}{3} \cdot \frac{n}{n+1} = \frac{|x-1|}{3} \cdot \lim_{n\to\infty}\frac{n}{n+1} = \frac{|x-1|}{3} \cdot 1,$$

we solve $|x-1|/3 < 1$ to find where the power series converges (by the ratio test). This yields $|x-1| < 3$ and thus the radius of convergence is $R = 3$. The power series converges for $-2 < x < 4$ and diverges for $|x-1| > 3$. However, we must check the endpoints $x = -2$ and $x = 4$ directly to determine if the series converges at these points.

Substituting $x = -2$ into the original series gives

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test (it is $-1$ times the Alternating Harmonic Series). Substituting $x = 4$ into the original series gives

$$\sum_{n=1}^{\infty} \frac{3^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges since it is the Harmonic Series (a $p$-series with $p = 1$). Therefore, the interval of convergence for the power series is $-2 \le x < 4$ or $[-2, 4)$.

3. $\displaystyle\sum_{n=1}^{\infty} (x-1)^n \frac{1}{n^2\,3^n}$

**Answer:** Notice the similarity with the previous problem. The center of the series is $c = 1$. Let $a_n = \frac{(x-1)^n}{n^2 3^n}$. We apply the ratio test regarding $x$ as some fixed value. We find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-1)^{n+1}}{(n+1)^2 3^{n+1}}}{\frac{(x-1)^n}{n^2 3^n}}\right| = \left|\frac{(x-1)^{n+1}}{(n+1)^2 3^{n+1}}\right| \cdot \left|\frac{n^2 3^n}{(x-1)^n}\right| = \frac{|(x-1)^n \cdot (x-1)|}{(n+1)^2 3^n \cdot 3} \cdot \frac{n^2 3^n}{|(x-1)^n|}$$

$$= \frac{|x-1|}{3} \cdot \frac{n^2}{n^2+2n+1}.$$

Then, since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-1|}{3} \cdot \frac{n^2}{n^2+2n+1} = \frac{|x-1|}{3} \cdot \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = \frac{|x-1|}{3} \cdot 1,$$

we solve $|x-1|/3 < 1$ to find where the power series converges (by the ratio test). This yields $|x-1| < 3$ and thus the radius of convergence is $R = 3$. The power series converges for $-2 < x < 4$ and diverges for $|x-1| > 3$. However, we must check the endpoints $x = -2$ and $x = 4$ directly to determine if the series converges at these points.

Substituting $x = -2$ into the original series gives

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges by the Alternating Series Test or by the Absolute Convergence Test. Substituting $x = 4$ into the original series gives

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2\,3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges since it is a $p$-series with $p = 2$. Therefore, the interval of convergence for the power series is $-2 \le x \le 4$ or $[-2, 4]$.