MATH 136-03 Calculus 2, Spring 2019

Section 10.6: Power Series

This section concerns infinite series where the terms being summed are functions of x, specifically power functions of the form $(x-c)^n$ for some constant c. These series, called power series, play an important role in applications of calculus since they are excellent approximations to more complicated functions such as e^x and $\sin x$.

Definition: Power Series

A **power series centered at** c is an infinite series of the form

$$F(x) \;=\; \sum_{n=0}^{\infty} a_n(x-c)^n \;=\; a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots .$$

Here, the **center** of the series is the constant c and the variable is x.

Example 1: The infinite series

$$F(x) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!}(x-3)^n \;=\; 1 + (x-3) + \frac{1}{2!}(x-3)^2 + \frac{1}{3!}(x-3)^3 + \frac{1}{4!}(x-3)^4 + \cdots \qquad (1)$$

is a power series centered at $c = 3$. Note that $0! = 1$ by convention. The series

$$G(x) \;=\; \sum_{n=1}^{\infty} \frac{1}{n}x^n \;=\; x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots \qquad (2)$$

is a power series centered at $c = 0$. Note that although this series begins at $n = 1$, it is still considered a power series.

The main question when studying power series is to determine the set of x-values for which the series converges. For example, in the series defined in equation (1) above, we have

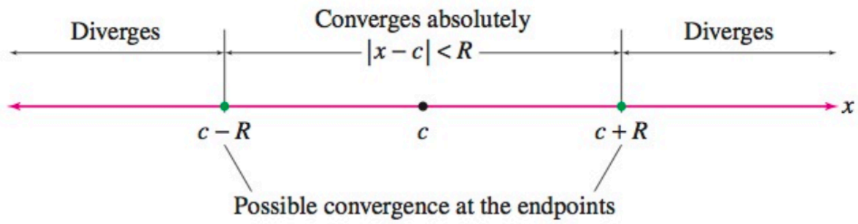
$$F(5) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!}(5-3)^n \;=\; \sum_{n=0}^{\infty} \frac{2^n}{n!},$$

which converges by the ratio test (see Example 1 on the worksheet for Section 10.5). This allows us to define the function F at $x = 5$ to be the unique number that the infinite series converges to. On the other hand, in the series defined in equation (2) above, we have

$$G(1) \;=\; \sum_{n=1}^{\infty} \frac{1}{n}(1)^n \;=\; \sum_{n=1}^{\infty} \frac{1}{n} \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots ,$$

which diverges because it is the Harmonic Series. Therefore, $G(1)$ is undefined.

While it may seem daunting to find the set of all x for which a given power series converges, it turns out that there is a unique value $R \geq 0$, called the **radius of convergence**, such that the power series converges absolutely for $|x-c| < R$ and diverges when $|x-c| > R$. In other words, for any power series centered at c, there is an **interval of convergence** centered at c of the form $c - R < x < c + R$



for which the power series converges. The series may or may not converge at the endpoints $x = c - R$ or $x = c + R$ (see figure above). If $R = 0$, then the series converges only when $x = c$. If $R = \infty$, then the power series converges for all x. The radius of convergence can be found using the ratio test.

Example 2: Use the ratio test to determine where $F(x) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}(x-3)^n$ converges.

Let $a_n = \frac{(x-3)^n}{n!}$. We apply the ratio test regarding x as some fixed value. We find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-3)^{n+1}}{(n+1)!}}{\frac{(x-3)^n}{n!}}\right| = \left|\frac{(x-3)^{n+1}}{(n+1)!}\right| \cdot \left|\frac{n!}{(x-3)^n}\right| = \frac{|(x-3)^n \cdot (x-3)|}{(n+1)\cdot n!} \cdot \frac{n!}{|(x-3)^n|} = \frac{|x-3|}{n+1}.$$

Then, since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-3|}{n+1} = |x-3|\cdot\lim_{n\to\infty}\frac{1}{n+1} = 0$, the power series converges for any x. The solution is $(-\infty,\infty)$ or $\mathbb{R}$. The radius of convergence is $R = \infty$.

Example 3: Use the ratio test to determine where $G(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}x^n$ converges.

Let $a_n = \frac{x^n}{n}$. We apply the ratio test regarding x as some fixed value. We find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \left|\frac{x^{n+1}}{n+1}\right| \cdot \left|\frac{n}{x^n}\right| = \frac{|x^n \cdot x|}{n+1} \cdot \frac{n}{|x^n|} = |x| \cdot \frac{n}{n+1}.$$

Then, since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}|x|\cdot\frac{n}{n+1} = |x|\cdot\lim_{n\to\infty}\frac{n}{n+1} = |x|$, the power series converges for any x satisfying $|x| < 1$ by the ratio test. The radius of convergence is $R = 1$. This shows that the power series converges for $-1 < x < 1$ and diverges for $|x| > 1$. However, we must check the endpoints $x = 1$ and $x = -1$ directly to determine if the series converges at these points. We have already seen that $G(1) = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges since it is the Harmonic Series. On the other hand, notice that

$$G(-1) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - + \cdots$$

converges by the Alternating Series Test (it is -1 times the Alternating Harmonic Series). We conclude that the power series $G(x)$ converges for $-1 \le x < 1$ or $[-1, 1)$.

Exercises: Find the radius R and interval of convergence for each of the following power series. Be sure to check the endpoints.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

2. $\displaystyle\sum_{n=1}^{\infty} (x-1)^n \frac{1}{n\,3^n}$

3. $\displaystyle\sum_{n=1}^{\infty} (x-1)^n \frac{1}{n^2\,3^n}$