MATH 136-03 Calculus 2, Spring 2019

Section 5.9: Compound Interest and Present Value

In this section we will focus on two important applications in finance that involve the exponential function e^x: **compound interest** and **present value**. There are other applications discussed in Section 5.9 (e.g., carbon dating and half-life), but we will not study those here.

Compound Interest

When you invest in a mutual fund or deposit your money into a bank account or CD, you expect to make a certain rate of return r on your money. Your money can be **compounded** at different times during the year. Compounded **yearly** means that after one year, you make r percent of your initial investment and add it to your original. For example, if you deposit \$1,000 at an interest rate of 5%, then after one year you will have

$$1{,}000 + 0.05 \cdot 1{,}000 \ = \ 1{,}000(1 + 0.05) \ = \ 1.05 \cdot 1{,}000 \ = \ \$1{,}050.$$

If you had invested \$2,000 instead, you would have

$$2{,}000 + 0.05 \cdot 2{,}000 \ = \ 2{,}000(1 + 0.05) \ = \ 1.05 \cdot 2{,}000 \ = \ \$2{,}100$$

at the end of the year. The new amount is simply 1.05 times the previous amount. Notice that the change in the amount in your account (\$50 in the first case, \$100 in the second) is proportional to the amount you invest, with the proportionality constant equal to the interest rate of 5%. The more you invest, the more you stand to earn.

Continuing with our example, we will have

$$1.05 \cdot 1{,}050 \ = \ (1.05)^2 \cdot 1{,}000 \ = \ \$1{,}102.50$$

after two years and

$$1.05 \cdot 1{,}102.50 \ = \ (1.05)^2 \cdot 1{,}050 \ = \ (1.05)^3 \cdot 1{,}000 \approx \$1{,}157.63$$

after three years. A clear pattern has emerged. The amount of money in the account after t years is simply

$$P(t) \ = \ 1000(1.05)^t, \tag{1}$$

an **exponential** function. This is the magic of compound interest. Your money grows at an exponential rate, rather than a linear one. If you only earned \$50 each year, you would have grown your account by \$150. However, because the money is compounded each year, your account has actually grown by \$157.63. This may seem like a small amount now, but it will have big consequences down the road.

If your money is compounded **quarterly**, the rate is adjusted to $r/4$, but your money is compounded 4 times a year (usually it is every 3 months). Compounded **monthly** means a rate of $r/12$ compounded 12 times a year while compounded **daily** means a rate of $r/365$ compounded every day of the year. In other words, the more your money is compounded, the lower the rate (seems bad), but the more often its compounded (seems good). So which is better, compounded yearly, quarterly, monthly, daily, or possibly continuously (every possible instant)?

Example 1: Suppose that $P_0 = \$1{,}000$ is invested at an annual rate of $r = 5\%$. Find the amount of money in the account at the end of one year if it is compounded annually, quarterly, monthly, or daily. Which yields the most money?

Answer: If the money is compounded annually, then there is $P_0 \cdot (1+r) = 1{,}000 \cdot 1.05 = \$1{,}050$ after one year. The value 1.05 is called the **yearly multiplier** because we multiply the current balance by this value to obtain the amount in the account after one year.

Compounded quarterly means the yearly multiplier changes to $(1+r/4)^4$ because the rate has been adjusted to $r/4$ but it is applied four times in a year (hence the 4 as an exponent). We obtain

$$P_0 \cdot (1+r/4)^4 \;=\; 1{,}000 \cdot (1+0.05/4)^4 \;\approx\; \$1{,}050.95.$$

To compound monthly, we use the multiplier $(1+r/12)^{12}$ since the rate becomes $r/12$ but it is applied 12 times in a year:

$$P_0 \cdot (1+r/12)^{12} \;=\; 1{,}000 \cdot (1+0.05/12)^{12} \;\approx\; \$1{,}051.16.$$

Finally, compounding daily means multiplying the initial amount by $(1+r/365)^{365}$ which gives

$$P_0 \cdot (1+r/365)^{365} \;=\; 1{,}000 \cdot (1+0.05/365)^{365} \;\approx\; \$1{,}051.27.$$

It appears that the more often we compound, the larger the account becomes. Even though the rate is decreased, we earn more money because the yearly multiplier increases.

In general, if an initial amount P_0 is deposited into an account earning interest at an annual rate r, compounded M times a year, then after t years the value of the account is

$$\boxed{P(t) \;=\; P_0 \left(1 + \frac{r}{M}\right)^{Mt}.}$$

The reason for the exponent Mt is that the **yearly multiplier** $(1+r/M)^M$ is multiplied by itself t times (once for each year). We then use the rule $(x^M)^t = x^{Mt}$.

Compounding Continuously

The best case scenario for growing our account would be to compound it at every possible instant. This is known as **compounding continuously**. Mathematically, we compute the yearly multiplier as $M \to \infty$. It turns out that

$$\lim_{M\to\infty} \left(1 + \frac{r}{M}\right)^M \;=\; e^r\,.$$

For a proof of this fact see Exercise 61 on p. 328 of the course textbook. Returning to our example above, we see that an initial investment of $P_0 = \$1{,}000$ at an annual rate of 5% compounded continuously yields

$$P_0 \cdot e^r \;=\; 1{,}000 \cdot e^{0.05} \;\approx\; \$1{,}051.27.$$

at the end of one year. This is just a fraction of one cent higher than if we had compounded daily. In general, if P_0 is deposited at an annual rate r compounded continuously, then after t years the value of the account is

$$\boxed{P(t) \;=\; P_0 e^{rt}.}$$

Exercise 1: Suppose that $P_0 = \$5{,}000$ is invested in an account paying at an annual rate of 7%. Find the amount in the account after 8 years if it is compounded (a) quarterly, (b) monthly, and (c) continuously.

Exercise 2: A bank pays interest at an annual rate of 3.5%. What is the yearly multiplier (to 6 decimal places) if the interest is compounded (a) 5 times a year, (b) 30 times a year, or (c) continuously?

Present Value

In business or finance it is often useful to know how much should be invested now to make a given amount of money at a later time. This is known as **present value**, abbreviated PV. For example, suppose we have an interest rate of 5% compounded continuously and we want to know how much money we have to invest now in order to have \$1,000 in 6 years. We must solve the equation

$$P_0 e^{0.05 \cdot 6} = 1{,}000$$

for P_0. The solution is $1{,}000/e^{0.05\cdot 6}$ or $1{,}000e^{-0.05\cdot 6} \approx \$740.82$. In financial terms, we say that the PV of \$1,000 received 6 years in the future is \$740.82. In general, we have

the PV of P dollars received t years in the future is Pe^{-rt}.

The present value of P is the amount you would have to invest *now* in order to have P dollars t years in the future.

Exercise 3: How much should you invest today in order to receive \$10,000 in 5 years if interest is compounded continuously at a rate of 2.5%?

Exercise 4: Is it better to receive \$500 today or \$600 in 5 years if the interest rate is 3%? What if the rate increases to 4%? Assume that interest is compounded continuously.

Exercise 5: Congratulations, you just won \$2 million dollars in the lottery! However, you do not get all of your money now; you will receive four yearly payments of \$500,000 beginning immediately. Assuming an interest rate of 5%, what is the present value of your prize? How much do you "lose" by not receiving the full prize today?

PV of an Income Stream

An **income stream** is a series of regular payments paid out over some length of time. For example, suppose that an investment pays you \$500 a year for 5 years. This means you will receive a total of \$2,500 dollars at the end of 5 years. However, the present value of your income stream is less than this amount. Assuming an interest rate of 4% and that payments are made at the end of the year, the PV of your income stream is

$$500e^{-0.04} + 500e^{-0.04\cdot 2} + 500e^{-0.04\cdot 3} + 500e^{-0.04\cdot 4} + 500e^{-0.04\cdot 5} \approx 2{,}220.85\,.$$

Can you explain the sum on the previous line?

Mathematically speaking, it is easier to assume the payments are made continuously rather than annually. This converts the sum above into an integral. The PV of an income stream paying out $R(t)$ dollars per year continuously for T years at an interest rate r is

$$\boxed{PV = \int_0^T R(t)e^{-rt}\,dt\,.}$$

The $R(t)\,dt \approx R(t)\,\Delta t$ term represents the amount paid out over a small time interval of length Δt. As Δt goes to 0, the Riemann sum becomes an integral.

Exercise 6: Find the PV of an income stream paying out continuously at a rate of \$750 per year for 10 years, assuming an interest rate of 5%.

Exercise 7: Find the PV of an investment that pays out continuously at a rate of $R(t) = \$1{,}000e^{0.03t}$ per year for 8 years, assuming an interest rate of 6%.