MATH 136-03 Calculus 2, Spring 2019

Section 5.8: Further Transcendental Functions

This section focuses on a few more integration formulas involving exponential and inverse trig functions. Here, the challenging part is not the new formulas, but rather making the correct u-substitution in order to convert the integrand into the required form.

The new integral formulas come from the corresponding formulas for the derivative:

1. $\displaystyle\int a^x \, dx = \frac{a^x}{\ln a} + c$ for any real number $a > 0$

2. $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c$

3. $\displaystyle\int \frac{1}{x^2+1} \, dx = \tan^{-1} x + c$

Note that in formula 1., if we choose $a = e$, then we obtain $\int e^x \, dx = e^x + c$ (as expected) because $\ln e = 1$.

Example 1: Evaluate $\displaystyle\int_0^{1/2} \frac{3}{\sqrt{1-x^2}} \, dx$.

Answer: Using formula 2., we have

$$\int_0^{1/2} \frac{3}{\sqrt{1-x^2}} \, dx \; = \; 3\sin^{-1}(x)\Big|_0^{1/2} \; = \; 3(\sin^{-1}(1/2) - \sin^{-1}(0)) \; = \; 3(\pi/6 - 0) \; = \; \pi/2 \, .$$

Example 2: Find $\displaystyle\int \frac{1}{9x^2+16} \, dx$ using the substitution $u = \dfrac{3x}{4}$.

Answer: The hard part here is that we cannot directly apply formula 3. because the denominator isn't in the correct form. The trick is to make a u-substitution so that the new integral can be evaluated with formula 3. Letting $u = \dfrac{3x}{4}$, we have $du = \frac{3}{4} \, dx$ or $dx = \frac{4}{3} \, du$. We also have

$$x \; = \; \frac{4u}{3} \quad \Longrightarrow \quad 9x^2 + 16 \; = \; 9\frac{16u^2}{9} + 16 \; = \; 16(u^2+1) \, .$$

Thus, in the u-variable, the integral transforms to

$$\int \frac{1}{16(u^2+1)} \cdot \frac{4}{3} \, du \; = \; \frac{1}{12} \int \frac{1}{u^2+1} \, du \; = \; \frac{1}{12} \tan^{-1} u + c \; = \; \frac{1}{12} \tan^{-1}\left(\frac{3x}{4}\right) + c \, .$$

In general, the following u-substitutions allow for successful applications of formulas 2 and 3:

- For integrals of the form $\displaystyle\int \frac{1}{\sqrt{a^2 - b^2x^2}} \, dx$, use $u = \dfrac{bx}{a}$.
- For integrals of the form $\displaystyle\int \frac{1}{a^2x^2 + b^2} \, dx$, use $u = \dfrac{ax}{b}$.

Exercises:

1. Evaluate $\displaystyle\int_0^1 \frac{4}{x^2+1}\,dx$.

2. Evaluate $\displaystyle\int_{-1/2}^{1/2} 9^t\,dt$

3. Evaluate $\displaystyle\int_0^{\frac{3}{2\sqrt{2}}} \frac{1}{\sqrt{9-4x^2}}$ using the substitution $u = \dfrac{2x}{3}$.

4. Evaluate $\displaystyle\int \frac{1}{\sqrt{1-36\theta^2}}\,d\theta$.

5. Evaluate $\displaystyle\int x^2 5^{x^3}\,dx$.

6. Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{x+1}{x^2+1}\,dx$. *Hint:* Split the integrand into two pieces.

7. Evaluate $\displaystyle\int \frac{x}{x^4+1}\,dx$. *Hint:* Use the substitution $u = x^2$.

8. Evaluate $\displaystyle\int \frac{\ln(\sin^{-1} x)}{(\sin^{-1} x)\sqrt{1-x^2}}\,dx$. *Hint:* Do two u-substitutions.