

MATH 136-03 Calculus 2, Spring 2019

Section 7.7: Improper Integrals

This section covers integrals that have ∞ or $-\infty$ (or both) as a limit of integration, as well as integrals where the integrand has a vertical asymptote at one of the limits of integration (or between them). These kinds of integrals are called **improper integrals**.

Definition: An **improper integral** is a definite integral where either

- (a) a limit of integration is ∞ or $-\infty$, or
- (b) the integrand approaches ∞ or $-\infty$ at a limit of integration or somewhere in between the limits of integration.

The following are all examples of improper integrals. The first two are of type (a) in the above definition, while the second two are of type (b).

$$\int_0^{\infty} e^{-2x} dx, \quad \int_{-\infty}^1 \frac{1}{x^2 + 1} dx, \quad \int_0^1 \frac{1}{x} dx, \quad \int_{-2}^2 \frac{1}{\sqrt{x+2}} dx.$$

In terms of area, improper integrals of type (a) are used to determine whether the area under a curve is finite, even if one of the limits of integration is infinite. Improper integrals of type (b) ask the same question except now the area is in the vertical direction (up against a vertical asymptote).

Here is the general approach for evaluating improper integrals:

1. Recognize that the definite integral is improper.
2. Replace the “bad” limit of integration with a variable b .
3. Find the antiderivative of the integrand and plug in b .
4. Take the limit of the result as b approaches the “bad” value. If this limit exists and is finite, we say the integral **converges**, if the limit does not exist, we say the integral **diverges**.

Example 1: A convergent integral

Consider the integral $\int_0^{\infty} e^{-2x} dx$. Notice that the integrand e^{-2x} approaches 0 as $x \rightarrow \infty$ (a decaying exponential). However, is the area under the curve finite or infinite as $x \rightarrow \infty$? Somewhat surprisingly, the area is indeed finite (it equals $1/2$). Even though $x \rightarrow \infty$, the integrand approaches 0 so fast that the area under the curve is finite. To see this, we compute

$$\begin{aligned} \int_0^{\infty} e^{-2x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-2x} dx \\ &= \lim_{b \rightarrow \infty} \left. -\frac{1}{2} e^{-2x} \right|_0^b \\ &= \lim_{b \rightarrow \infty} -\frac{1}{2} e^{-2b} + \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

Thus the integral converges to $1/2$.

Example 2: A divergent integral

Consider the integral $\int_1^{\infty} \frac{1}{x} dx$. As with Example 1, the integrand approaches 0 as $x \rightarrow \infty$.

However, this time the area under the curve is infinite. The function $f(x) = 1/x$ does not approach 0 fast enough to have finite area. In this case we have

$$\begin{aligned}\int_1^{\infty} \frac{1}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} \ln |x| \Big|_1^b \\ &= \lim_{b \rightarrow \infty} \ln b - 0 \\ &= \infty,\end{aligned}$$

because the graph of $\ln x$ goes to ∞ (slowly) as $x \rightarrow \infty$. Thus the integral diverges.

Example 3: A convergent integral of type (b)

Consider the integral $\int_0^1 \frac{1}{\sqrt{x}} dx$. This integral is improper because the integrand is undefined at $x = 0$ (vertical asymptote). However, the area under the curve is finite despite the fact that the integrand is going to ∞ as $x \rightarrow 0^+$. In this case, the integrand goes to ∞ *slow* enough for the area to be finite. To see this, we compute

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{b \rightarrow 0^+} \int_b^1 \frac{1}{\sqrt{x}} dx \\ &= \lim_{b \rightarrow 0^+} 2x^{1/2} \Big|_b^1 \\ &= \lim_{b \rightarrow 0^+} 2 - 2\sqrt{b} \\ &= 2.\end{aligned}$$

Thus the integral converges to 2.

Exercises: Determine whether each integral converges or diverges. If the integral converges, find the value of the integral.

1. $\int_{-1}^{\infty} \frac{1}{1+x^2} dx$

2. $\int_{\pi}^{\infty} \sin t dt$

3. $\int_0^{\infty} x e^{-4x} dx$

4. $\int_0^2 \frac{1}{x} dx$

5. $\int_3^{\infty} \frac{1}{x^2 - 1} dx$

6. $\int_0^4 \frac{1}{\sqrt{16 - x^2}} dx$