MATH 136-03 Calculus 2, Spring 2019

Section 7.1: Integration by Parts SOLUTIONS

1. Use integration by parts to compute $\int x \sin x \, dx$. Let $u = x$ and $dv = \sin x \, dx$.

   Answer: Since $u = x$ and $dv = \sin x \, dx$, we have $du = 1 \, dx$ and $v = -\cos x$. Using the integration by parts formula gives

   $$\int x \sin x \, dx \;=\; -x\cos x - \int -\cos x \, dx \;=\; -x\cos x + \int \cos x \, dx \;=\; -x\cos x + \sin x + c\,.$$

2. Compute $\int \ln x \, dx$ by setting $u = \ln x$ and $dv = 1 \, dx$.

   Answer: Since $u = \ln x$ and $dv = 1 \, dx$, we have $du = 1/x \, dx$ and $v = x$. Using the integration by parts formula gives

   $$\int \ln x \, dx \;=\; x\ln x - \int x \cdot \frac{1}{x} \, dx \;=\; x\ln x - \int 1 \, dx \;=\; x\ln x - x + c\,.$$

   Note that we simplify the expression in the integrand **before** trying to evaluate the integral.

3. Evaluate $\int x^3 \ln x \, dx\,.$

   Answer: Here we take $u = \ln x$ and $dv = x^3 \, dx$. Had we gone with $u = x^3$ and $dv = \ln x \, dx$, we would need to integrate $\ln x$ to find v, which is too complicated (see the previous problem). Thus $du = 1/x \, dx$ and $v = x^4/4$. Using the integration by parts formula gives

   $$\int x^3 \ln x \, dx \;=\; \frac{x^4}{4}\ln x - \int \frac{x^4}{4} \cdot \frac{1}{x} \, dx \;=\; \frac{x^4}{4}\ln x - \int \frac{x^3}{4} \, dx$$

   $$=\; \frac{x^4}{4}\ln x - \frac{x^4}{16} + c \;=\; \frac{x^4}{16}\,(4\ln x - 1) + c\,.$$

4. Evaluate $\int x^2 \sin(2x) \, dx\,.$

   Answer: This problem involves integration by parts twice. First, let $u = x^2$ and $dv = \sin(2x) \, dx$. Then $du = 2x \, dx$ and $v = -\frac{1}{2}\cos(2x)$. Using the integration by parts formula gives

   $$\int x^2 \sin(2x) \, dx \;=\; -\frac{x^2}{2}\cos(2x) - \int -\frac{1}{2} \cdot 2x\cos(2x) \, dx \;=\; -\frac{x^2}{2}\cos(2x) + \int x\cos(2x) \, dx\,.$$

   The remaining integral also requires integration by parts, but notice that the power has dropped from 2 in the original to 1. We let $u = x$ and $dv = \cos(2x) \, dx$. Then $du = 1 \, dx$ and $v = \frac{1}{2}\sin(2x)$. Therefore

   $$\int x\cos(2x) \, dx \;=\; \frac{x}{2}\sin(2x) - \int \frac{1}{2}\sin(2x) \, dx \;=\; \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + c\,.$$

   Substituting this result into the previous line, we find

   $$\int x^2 \sin(2x) \, dx \;=\; -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + c\,.$$

5. Evaluate $\displaystyle\int_0^1 \tan^{-1} x \, dx$.

Answer: Let $u = \tan^{-1} x$ and $dv = 1\, dx$. Then $du = \frac{1}{x^2+1}\, dx$ and $v = x$. Using the integration by parts formula gives

$$\int_0^1 \tan^{-1} x \, dx \;=\; x \tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\, dx \;=\; \frac{\pi}{4} - \int_0^1 \frac{x}{x^2+1}\, dx\,,$$

since $\tan^{-1}(1) = \pi/4$. The remaining integral can be done using a u-sub with $u = x^2+1$ and $du = 2x\, dx$. Also, if $x = 0$, then $u = 1$ and if $x = 1$, then $u = 2$. We have

$$\int_0^1 \frac{x}{x^2+1}\, dx \;=\; \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\, dx \;=\; \frac{1}{2}\int_1^2 \frac{1}{u}\, du \;=\; \frac{1}{2}\ln u\Big|_1^2 \;=\; \frac{1}{2}\ln 2\,.$$

Therefore the integral evaluates to $\dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$.

6. Evaluate $\displaystyle\int e^{3x}\cos x\, dx$.

Hint: Let $I = \displaystyle\int e^{3x}\cos x\, dx$. Do integration by parts twice and then solve for I.

Answer: First we let $u = e^{3x}$ and $dv = \cos x\, dx$. Then $du = 3e^{3x}\, dx$ and $v = \sin x$. Using the integration by parts formula gives

$$\int e^{3x}\cos x\, dx \;=\; e^{3x}\sin x - \int 3e^{3x}\sin x\, dx\,.$$

Next we do integration by parts again, letting $u = 3e^{3x}$ and $dv = \sin x\, dx$. Then $du = 9e^{3x}\, dx$ and $v = -\cos x$. Thus

$$\int 3e^{3x}\sin x\, dx \;=\; -3e^{3x}\cos x - \int -9e^{3x}\cos x\, dx \;=\; -3e^{3x}\cos x + 9\int e^{3x}\cos x\, dx\,.$$

At this point it may feel like we are going around in circles given that we have arrived back at the original integral. However, if we let $I = \displaystyle\int e^{3x}\cos x\, dx$, then we have discovered that

$$I \;=\; e^{3x}\sin x - \left(-3e^{3x}\cos x + 9I\right) \;=\; e^{3x}\sin x + 3e^{3x}\cos x - 9I\,.$$

Solving for I we find

$$10I \;=\; e^{3x}\sin x + 3e^{3x}\cos x \;=\; e^{3x}(\sin x + 3\cos x) \quad\Longrightarrow\quad I \;=\; \frac{1}{10}e^{3x}(\sin x + 3\cos x) + c\,.$$

7. Evaluate $\displaystyle\int \cos(\sqrt{x}\,)\, dx$. *Hint:* First let $u = \sqrt{x}$ and do a u-sub. Then do integration by parts.

Answer: Let $u = \sqrt{x}$. Then $du = \frac{1}{2}x^{-1/2}\, dx = \frac{1}{2\sqrt{x}}\, dx = \frac{1}{2u}\, dx$. This in turn gives $dx = 2u\, du$ so the integral transforms as

$$\int \cos(\sqrt{x}\,)\, dx \;=\; \int \cos(u)\cdot 2u\, du \;=\; 2\int u\cos u\, du\,.$$

This last integral can be done using integration by parts. Let $w = u$ and $dv = \cos u \, du$. Then $dw = 1 \, du$ and $v = \sin u$. The integration by parts formula gives

$$2\int u\cos u \, du \;=\; 2\left(u\sin u - \int \sin u \, du\right) \;=\; 2\,(u\sin u + \cos u) + c\,.$$

Returning to the original variable, we have

$$\int \cos(\sqrt{x}\,)\, dx \;=\; 2\left(\sqrt{x}\,\sin(\sqrt{x}\,) + \cos(\sqrt{x}\,)\right) + c\,.$$