MATH 136-03 Calculus 2, Spring 2019

Section 7.3: Trigonometric Substitution

This section focuses on integrals involving expressions of the form $\sqrt{a^2 - x^2}$ or $\sqrt{x^2 + a^2}$, where a is a constant. For these types of integrals, the key is to make a **trig substitution** that converts the integral into a simpler form. Then, after computing the integral in the new variable (usually θ), we use a right triangle and SOH-CAH-TOA to convert back into the original variable.

Useful Trig Identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad (1)$$

$$1 + \tan^2\theta = \sec^2\theta \qquad (2)$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad (3)$$

Note that identity (2) above can be derived from (1) by dividing both sides of the equation by $\cos^2\theta$.

The general technique of trig substitution is:

- for integrals containing $\sqrt{a^2 - x^2}$, use the substitution $x = a\sin\theta$,
- for integrals with $\sqrt{x^2 + a^2}$, or $x^2 + a^2$ in the denominator, use the substitution $x = a\tan\theta$.

Integrals involving $\sqrt{a^2 - x^2}$

Example 1:

Compute $\int \sqrt{9 - x^2}\, dx$ using the substitution $x = 3\sin\theta$.

Answer: Letting $x = 3\sin\theta$, we have $dx = 3\cos\theta\, d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9(1 - \sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta.$$

Thus, the integral transforms to

$$\begin{aligned}\int 3\cos\theta \cdot 3\cos\theta\, d\theta &= 9\int \cos^2\theta\, d\theta \\ &= 9\int \frac{1}{2}\left(1 + \cos(2\theta)\right)\, d\theta \\ &= \frac{9}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + c \\ &= \frac{9}{2}(\theta + \sin\theta\cos\theta) + c\,.\end{aligned}$$

To finish the problem, we return to the original variable x. We have $\sin\theta = x/3$, so $\theta = \sin^{-1}(x/3)$. Using a right triangle with one angle equal to θ, we find $\cos\theta = \frac{1}{3}\sqrt{9 - x^2}$. Hence, the solution is

$$\frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{3}\cdot\frac{\sqrt{9 - x^2}}{3}\right) = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}x\sqrt{9 - x^2} + c\,.$$

Integrals involving $\sqrt{x^2+a^2}$

Example 2: Evaluate $\displaystyle\int_0^3 \frac{1}{\sqrt{x^2+16}}\,dx$ using the substitution $x = 4\tan\theta$.

Answer: Letting $x = 4\tan\theta$, we have $dx = 4\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+16} = \sqrt{16\tan^2\theta+16} = \sqrt{16(1+\tan^2\theta)} = \sqrt{16\sec^2\theta} = 4\sec\theta.$$

Moreover, since $\tan\theta = x/4$, we have $x = 0$ implies $\tan\theta = 0$ which means $\theta = 0$, and $x = 3$ implies $\tan\theta = 3/4$ yielding $\theta = \tan^{-1}(3/4)$. Thus, the integral transforms to

$$\begin{aligned}
\int_0^{\tan^{-1}(3/4)} \frac{1}{4\sec\theta}\cdot 4\sec^2\theta\,d\theta &= \int_0^{\tan^{-1}(3/4)} \sec\theta\,d\theta \\
&= \ln|\sec\theta+\tan\theta|\Big|_0^{\tan^{-1}(3/4)} \\
&= \ln|\sec(\tan^{-1}(3/4))+3/4| - \ln|1+0| \\
&= \ln|5/4+3/4| \\
&= \ln 2\,.
\end{aligned}$$

Here we evaluate $\sec(\tan^{-1}(3/4)$ by drawing a right triangle with angle θ and opposite side 3, adjacent side 4. Then $\sec\theta = 5/4$.

Exercises: Complete the following on a **separate** piece(s) of paper.

1. Evaluate $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$.

   Check your answer by interpreting the integral as the area under the curve.

2. Evaluate $\displaystyle\int \frac{1}{x(x^2+4)}\,dx$ using the substitution $x = 2\tan\theta$.

3. Evaluate $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$.

4. Evaluate $\displaystyle\int_{\sqrt{3}}^3 \frac{1}{x^2\sqrt{x^2+9}}\,dx$.