MATH 241-02, Multivariable Calculus, Spring 2019

Section 10.3: Arc Length and Curvature

This section describes how to calculate some geometric properties of a space curve such as its arc length and curvature. Some of the concepts in this section are complicated so be sure to read carefully. This material marks the beginning of an interesting branch of mathematics called **differential geometry.**

Arc Length

Imagine taking a piece of string and placing it along a space curve between two points P $(t = a)$ and Q $(t = b)$. The arc length of the curve between P and Q is equal to the length of the string. To find it with calculus, we sum up tiny distances along the curve using the tangent vector $\mathbf{r}'(t)$. Recall that $|\mathbf{r}'(t)|$ represents the speed at which the curve is traced out. Since distance equals speed $\times$ time, $|\mathbf{r}'(t)| \cdot \Delta t$ is the length of a tiny portion of the curve. To find the total length, we form a Riemann sum and take the limit as $\Delta t \to 0$. This results in the following integral formula for the length of a space curve.

Arc Length: If $\mathbf{r}(t) = < x(t),\, y(t),\, z(t) >$, where x, y, and z are differentiable functions, then the length of the space curve traced out by $\mathbf{r}(t)$ over the interval $a \leq t \leq b$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt\,.$$

Example 1: Find the length of one turn of the helix $\mathbf{r}(t) = <\cos t,\, \sin t,\, t>$ from $t = 0$ to $t = 2\pi$.

Answer: We have $\mathbf{r}'(t) = <-\sin t,\, \cos t\,,\, 1>$ so that $|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$. Then by the arc length formula, we obtain

$$L = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\, t\Big|_0^{2\pi} = 2\sqrt{2}\,\pi\,.$$

Exercise 1: Suppose that $\mathbf{r}(t) = < \cos(t^2),\, \sin(t^2),\, t^2 >$ from $t = 0$ to $t = \sqrt{2\pi}$. What curve is traced out by this parametrization? Find the arc length of the curve and compare your answer with Example 1. Explain.

Assuming you did the previous problem correctly, notice that different parametrizations can produce the *same* curve. The speed traveled along the curve may be different (in Example 1 it is $\sqrt{2}$ while in Exercise 1 it is $\sqrt{8}\, t$), but the shape and length of the curve are the same. The arc length L of a curve is **independent of the parametrization**, a feature we would expect to have since the length of the curve should be the same no matter how it is traced out.

Parametrization with respect to arc length

One of the most useful parametrizations of a curve is to use arc length as the time unit. In other words, traveling 4 units of "time" would mean traveling 4 units of length along the curve. The standard parameter for arc length is s. Below we demonstrate how to reparametrize a curve with respect to its arc length s. This can always be done in theory, but is rarely computable in practice.

We start by defining the arc length function

$$s(t) \;=\; \int_a^t |\mathbf{r}'(u)|\; du\,.$$

Here the variable is t, a is a constant (the starting time) and u is a "dummy" variable. The function $s(t)$ gives the length of the curve at time t. For example, $s(b) = 7$ means the curve is 7 units long from $t = a$ to $t = b$. Note that $s(a) = 0$ (no length). By the Fundamental Theorem of Calculus part 2,

$$\frac{ds}{dt} \;=\; |\mathbf{r}'(t)|\,. \tag{1}$$

This equation states that the rate of change of the arc length is equal to the speed traveled along the curve. Equation (1) is a differential equation (ODE) that sometimes can be solved to find $s(t)$. To **parameterize a curve with respect to arc length**, we invert the function $s(t)$ to find $t(s)$. The next example shows how this is accomplished for the helix.

Example 2: Reparametrize the helix $\mathbf{r}(t) =< \cos t,\, \sin t,\, t >$ with respect to arc length measured from the starting point $t = 0$.

Answer: Recall that $|\mathbf{r}'(t)| \;=\; \sqrt{(-\sin t)^2 + \cos^2 t + 1} \;=\; \sqrt{2}$. Thus we need to solve the ODE $\frac{ds}{dt} = \sqrt{2}\,$. Using the separate and integrate technique, we have

$$\frac{ds}{dt} \;=\; \sqrt{2} \quad\Longrightarrow\quad ds \;=\; \sqrt{2}\, dt \quad\Longrightarrow\quad s \;=\; \sqrt{2}\, t + c\,.$$

Since we are measuring arc length from $t = 0$, we know that $s = 0$ when $t = 0$. Plugging in these values to the last equation gives $c = 0$. Thus $s = \sqrt{2}\, t$, which in turn gives $t = \frac{1}{\sqrt{2}} s$. Replacing t by $\frac{1}{\sqrt{2}} s$ in the original parametrization gives a parametrization of the helix with respect to arc length:

$$\mathbf{r}(s) =< \cos\left(\frac{1}{\sqrt{2}} s\right),\, \sin\left(\frac{1}{\sqrt{2}} s\right),\, \left(\frac{1}{\sqrt{2}} s\right) > \;. \tag{2}$$

Exercise 2: Check that the arc length of the helix parametrized by equation (2) from $t = 0$ to $t = s$ is simply $L = s$.

Punchline: Reparametrizing a curve with respect to arc length changes the speed to a constant 1 (unit speed) so that distance = time = s.

Curvature

How do we measure how much a space curve "curves"? Recall from Calc 1 that the second derivative of a function is used to measure the concavity. For a space curve we measure the magnitude of the change in the unit tangent vector $\mathbf{T}$ with respect to arc length s. Since $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ is a first derivative, this generalizes the idea of concavity from Calc 1. We must differentiate with respect to arc length so that the curvature is independent of the parametrization.

The **curvature** of a space curve $\mathbf{r}(t)$ is

$$\kappa(s) \; = \; \left| \frac{d\mathbf{T}}{ds} \right| ,$$

where $\mathbf{T}$ is the unit tangent vector and s is the arc length parameter.

Note that curvature is not usually constant; it depends on the point that you are evaluating at. The more the tangent vector is changing at a point, the higher the curvature.

To find a formula for curvature that depends on t instead of s, we use the chain rule:

$$\frac{d\mathbf{T}}{dt} \; = \; \frac{d\mathbf{T}}{ds} \cdot \frac{ds}{dt} \qquad \Longrightarrow \qquad \kappa \; = \; \left| \frac{d\mathbf{T}}{ds} \right| \; = \; \frac{|d\mathbf{T}/dt|}{|ds/dt|} \; = \; \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

using equation (1). The formula $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|$ is easier to use because we do not need to reparametrize the curve with respect to arc length.

Exercise 3: Show that the curvature of a circle of radius ρ is $1/\rho$. Interpret your answer by drawing a few circles with different radii. *Hint:* Use the parametrization $\mathbf{r}(t) = \rho \cos t \, \mathbf{i} + \rho \sin t \, \mathbf{j}$.

Exercise 4: What is the curvature of a line? Confirm your guess by using the parametrization $\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{v}$.

Another formula for curvature, which is derived on pp. 710–711 of the textbook, is

$$\kappa(t) \; = \; \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \, .$$

The top of the fraction is the magnitude of the cross product of the first and second derivatives of $\mathbf{r}(t)$. This formula is often more convenient to apply because it avoids ugly quotient rule calculations.

Normal and Binormal Vectors

The goal of our final topic is to find three unit vectors that form an orthogonal frame describing the motion of a space curve. The unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are an example of an orthogonal frame. The three vectors we will use are the unit tangent vector $\mathbf{T}$, a unit normal vector $\mathbf{N}$, and a binormal vector $\mathbf{B}$. Together they form the **TNB frame**.

1. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ = unit tangent vector (direction of motion)
2. $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ = unit normal vector (direction curve is turning; assumes curvature is non-zero)
3. $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ = binormal vector (orthogonal to the plane containing $\mathbf{T}$ and $\mathbf{N}$, called the **osculating plane**)

It is clear that the vectors $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$ are all unit length, and that $\mathbf{B}$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$, but why is $\mathbf{N}$ orthogonal to $\mathbf{T}$? To see this we use the product rule for the dot product. Since $\mathbf{T}(t)$ has unit length for all time, we have $\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$ for all t. Differentiating both sides of this equation with respect to t gives

$$\mathbf{T}'(t) \cdot \mathbf{T}(t) + \mathbf{T}(t) \cdot \mathbf{T}'(t) \;=\; 0 \quad \Longrightarrow \quad \mathbf{T}(t) \cdot \mathbf{T}'(t) \;=\; 0\,,$$

which shows that $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal for all t. Since $\mathbf{N}$ is a scalar multiple of $\mathbf{T}'$, we see that $\mathbf{N}$ and $\mathbf{T}$ are always perpendicular, as desired.

Exercise 5: Find $\kappa(t)$, $\mathbf{T}(t), \mathbf{N}(t)$, and $\mathbf{B}(t)$ for the standard helix $\mathbf{r}(t) = <\cos t,\, \sin t,\, t>$.

We summarize the key formulas for this section below.

$$\frac{ds}{dt} = |\mathbf{r}'(t)|, \quad L = \int_a^b |\mathbf{r}'(t)|\, dt\,.$$

$$\kappa \;=\; \left|\frac{d\mathbf{T}}{ds}\right| \;=\; \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \;=\; \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}, \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}, \quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$