MATH 241-02, Multivariable Calculus, Spring 2019

Section 10.2: Derivatives and Integrals of Vector Functions

This section introduces the derivative and integral of a space curve as well as the unit tangent vector. The derivative of a space curve is straight-forward; just differentiate each component of the function. The same is true when computing the integral.

Derivatives

The definition of the derivative of a vector function $\mathbf{r}(t)$ is the same as the one for a real-valued function. It involves a small scalar h heading to 0 and a difference quotient that represents the slope of a secant vector, just as with Calc 1. In the limit, provided it exists, the secant vector approaches the **tangent vector** $\mathbf{r}'(t)$. Using the fact that the limit of a vector-valued function is the limit of each component, we see that the derivative of $\mathbf{r}(t)$ is simply the derivative of each component.

Derivative: If $\mathbf{r}(t) = <x(t),\, y(t),\, z(t)>$, where x, y, and z are differentiable functions, then

$$\mathbf{r}'(t) \;= <x'(t),\, y'(t),\, z'(t)>$$

and is called the **tangent vector**. The **unit tangent vector**, denoted $\mathbf{T}(t)$, is given by

$$\mathbf{T}(t) \;=\; \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\,.$$

Example 1: Consider the vector function $\mathbf{r}(t) = <\cos t,\, \sin t\,,\, t>$, which parametrizes the standard helix. Find **(a)** $\mathbf{r}'(t)$ and **(b)** the unit tangent vector at time $t = \pi$.

Answer: **(a)** Differentiating each component, we have $\mathbf{r}'(t) = <-\sin t,\, \cos t\,,\, 1>$.

(b) To find the unit tangent vector at $t = \pi$, we first compute $\mathbf{r}'(\pi) = <0,\, -1\,,\, 1>$. Then, to make this vector unit length, we divide by its length $|\mathbf{r}'(\pi)| = \sqrt{2}$. Thus $\mathbf{T}(\pi) = <0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}>$.

For this example, note that $|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$. This means that the helix is traced out at a constant **speed** of $\sqrt{2}$.

Exercise 1: Let $\mathbf{r}(t) = e^{6t}\,\mathbf{i} + t^2 \sin t\,\mathbf{j} + \tan(3t)\,\mathbf{k}$.

(a) Find $\mathbf{r}'(t)$.

(b) Find the unit tangent vector at $t = 0$.

Exercise 2: Find parametric equations for the tangent line to the space curve

$$\mathbf{r}(t) \;= < te^{-t},\, \sqrt{3-2t},\, \ln(t+4) >$$

at the point where $t = 0$.

Integration

Just like the derivative, the integral of a vector function is obtained by integrating each component. The result of integrating $\mathbf{r}(t)$ is a vector, although the geometric meaning of this vector is not clear.

Integration: If $\mathbf{r}(t) = < x(t),\, y(t),\, z(t) >$, then

$$\int \mathbf{r}(t)\, dt \;= < \int x(t)\, dt,\, \int y(t)\, dt,\, \int z(t)\, dt > \,.$$

Exercise 3: If $\mathbf{r}(t) = e^{6t}\,\mathbf{i} + \dfrac{t}{1+t^2}\,\mathbf{j} + \sin(\pi t)\,\mathbf{k}$, find $\displaystyle\int_0^1 \mathbf{r}(t)\, dt$.