MATH 241-02, Multivariable Calculus, Spring 2019

Section 10.1: Vector Functions and Space Curves

In this chapter we explore vector functions, where the input variable is one-dimensional (time t) and the output is a vector in two or more dimensions. Such a function will trace out a curve in the plane or in space. In this section we investigate the types of curves that are traced out by vector functions and begin to study the calculus of these functions.

Vector-Valued Functions

A **vector-valued function** or **vector function** for short, is a function from $\mathbb{R}$ to $\mathbb{R}^2$ or $\mathbb{R}^3$. The input variable is typically t and the output is a vector $\mathbf{r}(t)$. The domain of $\mathbf{r}(t)$ is the set of all t-values that can be plugged into **all** components of the function.

Exercise 1: Find the domain of the vector function $\mathbf{r}(t) = <\sin t, \sqrt{t+3}, \dfrac{4t}{t^2-25}>$.

One of the main goals with a vector-valued function is to determine what curve is traced out as t varies. We typically focus on just the head of the vector. Here is a simple example of a vector function in the plane.

Example 1: Sketch and describe the curve traced out by the vector function $\mathbf{r}(t) = <t, -t^2>$.

Answer: This is a vector function whose domain is $\mathbb{R}$ because there are no restrictions on the value of t. The output of the function is a subset of $\mathbb{R}^2$, the xy-plane. The component functions are $x(t) = t$ and $y(t) = -t^2$, which can be regarded as **parametric equations** with t as the parameter. One of the simplest ways to determine the curve traced out over time is to plot the points $(x(t), y(t))$ for different t-values, as demonstrated in the table below.

t	x	y
-3	-3	-9
-2	-2	-4
-1	-1	-1
0	0	0
1	1	-1
2	2	-4
3	3	-9

The points above lie on the parabola $y = -x^2$. As t increases we move from left to right along the parabola. For $t < 0$, we are in the third quadrant moving up towards the vertex at the origin, while for $t > 0$, we are in the fourth quadrant heading away from the vertex. Note that $y(t) = -(x(t))^2$ for any value of t since $-t^2 = -(t)^2$. This gives an algebraic justification that each point of the function lies on the curve $y = -x^2$.

Example 2: Sketch and describe the curve traced out by the vector function $\mathbf{r}(t) = <\cos t,\ \sin t,\ t>$.

Answer: This is a vector-valued function in $\mathbb{R}^3$ with domain $\mathbb{R}$. It is a well-known example of a **space curve**, a curve traced out in three dimensions. The table below hints at the underlying structure of the curve.

t	x	y	z
0	1	0	0
$\pi/2$	0	1	$\pi/2$
π	-1	0	π
$3\pi/2$	0	-1	$3\pi/2$
2π	1	0	2π
$5\pi/2$	0	1	$5\pi/2$
3π	-1	0	3π

Note that $x = \cos t$ and $y = \sin t$ are each periodic functions of period 2π, although $z = t$ is not; it keeps increasing with t. Also notice that $x(t)^2 + y(t)^2 = \cos^2 t + \sin^2 t = 1$ for all t so that the curve traced out always lies on the cylinder $x^2 + y^2 = 1$. The curve is called a **helix** (slinky) and spirals upward at a constant rate, going around the cylinder once every 2π time units (see Figure 1).

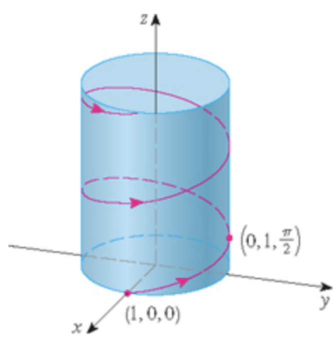

Figure 1: The curve traced out by $\mathbf{r}(t) = <\cos t, \sin t, t>$ is a helix that lies on the cylinder $x^2 + y^2 = 1$.

Exercise 2: Explain the difference between the helix in Example 2 and each of the following space curves: $\mathbf{r_1}(t) = <\cos(2t),\ \sin(2t),\ t>$ and $\mathbf{r_2}(t) = <\cos t,\ \sin t,\ e^t>$.

Exercise 3: What curve does the vector function $\mathbf{r_1}(t) = <3 - t,\ 4 + 2t,\ \pi - 5t>$ trace out in $\mathbb{R}^3$?

Exercise 4: Show that the curve traced out by $\mathbf{r}(t) = < \cos t \, \sin(2t), \, \sin t \, \sin(2t), \, \cos(2t) >$ lies on the unit sphere.

Limits of Vector Functions

The limit of a vector-value function is defined by taking the limit of each component, assuming that each limit exists. Note that the limit is also a vector.

Limits: If $\mathbf{r}(t) = < x(t), \, y(t), \, z(t) >$, then

$$\lim_{t \to a} \mathbf{r}(t) \; = < \lim_{t \to a} x(t), \, \lim_{t \to a} y(t), \, \lim_{t \to a} z(t) > ,$$

provided the limit of each component function exists.

The vector function $\mathbf{r}(t)$ is **continuous at a** provided that

$$\lim_{t \to a} \mathbf{r}(t) \; = \; \mathbf{r}(a) \, .$$

This generalizes the usual definition of continuity for a function of one variable. A continuous vector-valued function will trace out a space curve without any breaks or holes. It can be drawn without lifting your pencil off the page.

Exercise 5: Consider the vector function $\mathbf{r}(t) = e^{2t} \, \mathbf{i} + \dfrac{\sin(3t)}{t} \, \mathbf{j} + \ln(t+5) \, \mathbf{k}$.

(a) Find the domain of $\mathbf{r}(t)$.

(b) Find $\lim_{t \to 0} \mathbf{r}(t)$.

(c) Is $\mathbf{r}(t)$ continuous at $t = 0$? Explain.