MATH 241-02, Multivariable Calculus, Spring 2019

Section 11.5: The Chain Rule

The goal of this section is to generalize the chain rule from functions of one variable to those with several variables.

The Chain Rule and Tree Diagrams

The easiest way to generalize the chain rule is by using Leibniz notation. Recall from Calc 1 that if $y = f(x(t))$ is a composition of two functions, that is, y is a function of x and x is a function of t, then the derivative of y with respect to t is given by

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

(imagine the dx's canceling out). If $z = f(x(t), y(t))$ is a differentiable function of x and y, and x and y are each differentiable functions of t, then dz/dt is given by

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.}$$

This may also be written as

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.}$$

The idea behind the formula is to measure the change in both the x- and y-directions and then add them together. Notice how the dx and dy's cancel with the ∂x and ∂y's, respectively. Also note that we use the ∂ sign when the function has more than one variable, but we use d when there is only one variable.

A useful way to visualize the chain rule is to draw a **tree diagram**, as shown below. To compute dz/dt, we multiply the derivatives in each "chain" and then add them together.

$$\begin{tikzcd} & z \arrow[dl, dash, "f_x"'] \arrow[dr, dash, "f_y"] & \\ x \arrow[d, dash, "\frac{dx}{dt}"'] & & y \arrow[d, dash, "\frac{dy}{dt}"] \\ t & & t \end{tikzcd}$$

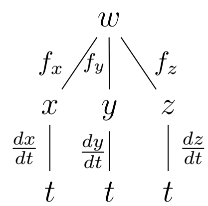
A tree diagram for visualizing the chain rule for $z = f(x(t), y(t))$.

Exercise 1: Suppose that $z = f(x, y) = y^2 + \sin(xy)$ and $x(t) = t^2 + 1$, $y(t) = 3t$. Compute dz/dt by **(a)** using the chain rule and **(b)** by substituting $x(t)$ and $y(t)$ into f and computing dz/dt directly.

The Chain Rule for Functions of More Variables

Example 1: Suppose that $w = f(x, y, z)$ and that $x = x(t), y = y(t)$, and $z = z(t)$. Draw a tree diagram and find a formula for dw/dt.

Answer:



A tree diagram for visualizing the chain rule for $w = f(x(t), y(t), z(t))$.

Based on the diagram, we have

$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}\,.}$$

Exercise 2: Suppose that $z = f(x, y)$ is a differentiable function of x and y, and that $x = x(u, v)$ and $y = y(u, v)$ are differentiable functions of u and v. Draw a tree diagram for z and use it to find formulas for $\partial z/\partial u$ and $\partial z/\partial v$. *Hint:* The formula for $\partial z/\partial u$ should not have any v's in it.

Exercise 3: Suppose that $z = y^2 e^x$, where $x = \ln u$ and $y = u - v$. Compute $\partial z/\partial u$ and $\partial z/\partial v$ by **(a)** using the chain rule and **(b)** by substituting x and y into z and computing the partials directly. Check that your answers agree.