MATH 241-02, Multivariable Calculus, Spring 2019

Section 11.7: Maximum and Minimum Values

In this section we learn how to find extrema (min's and max's) of a function of several variables. We also discuss the second derivative test for determining the type of critical point. The big change from functions of a single variable is the addition of a new kind of critical point, a saddle point.

Local min's and max's

Recall from Calc 1 that a function $f(x)$ has a local maximum at $x = a$ if the function values in a neighborhood of a are all smaller than $f(a)$. Likewise, $f(x)$ has a local minimum at $x = a$ if the function values in a neighborhood of a are all larger than $f(a)$. An absolute or global min or max is where the function has its lowest or highest value over the entire domain. It is important to remember that a function may have its global max or min at an endpoint (on the boundary of the domain).

All of the above theory transfers to functions of more than one variable.

Definition: $f(x, y)$ has a **local maximum** at (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) in a neighborhood of (a, b). $f(x, y)$ has a **local minimum** at (a, b) if $f(a, b) \leq f(x, y)$ for all (x, y) in a neighborhood of (a, b).

For functions of two variables, a **neighborhood** of a point is an open disk about the point (open means not including the boundary). In three variables a neighborhood is an open sphere; in higher dimensions, it is an open ball. When we are talking about **local** extrema, we always mean nearby, not over the entire domain.

Example 1: $f(x, y) = x^2 + y^2$

Recall that the graph of f is a bowl opening upwards with bottom resting at the origin (see Figure 1). Thus, f has a local minimum at $(0, 0)$. For any other point, there is always a direction which will decrease the value of f (follow $-\nabla f = -2 <x, y>$ toward the origin). In fact, f has an absolute minimum of 0 at the point $(0, 0)$ because

$$f(0,0) \;=\; 0 \;\leq\; f(x,y) \;=\; x^2+y^2 \quad \text{is true} \quad \forall (x,y) \in \mathbb{R}^2.$$

There are no local or absolute max's for f on $\mathbb{R}^2$.

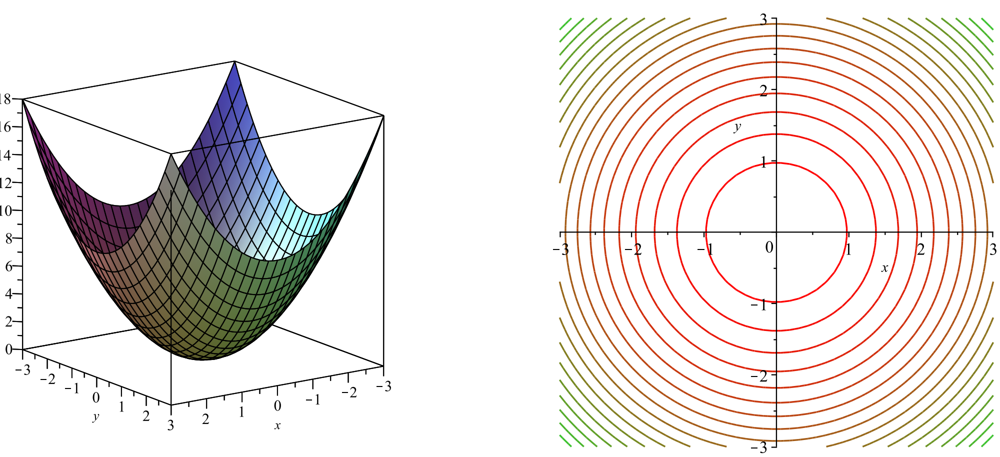

Figure 1: The function $f(x, y) = x^2 + y^2$ has an absolute minimum of 0 at the oirgin.

Example 2: $g(x, y) = 1 - x^2 - y^2$

In this case, the graph of g is a bowl opening downwards (see Figure 2). Here g has a local and absolute maximum of 1 at the point $(0, 0)$. For any other point, there is always a direction which will increase the value of g (follow $\nabla g = -2 < x, y >$ toward the origin). There are no local or absolute min's for g on $\mathbb{R}^2$.

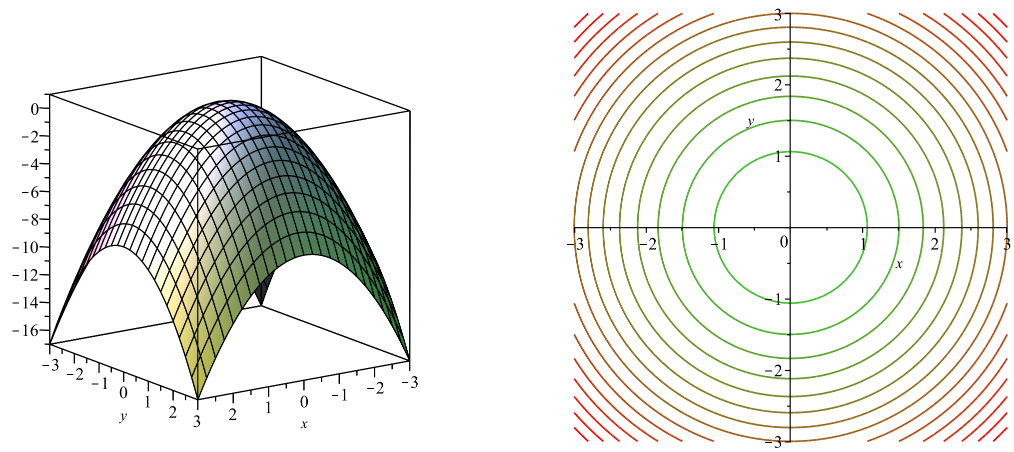

Figure 2: The function $g(x, y) = 1 - x^2 - y^2$ has an absolute maximum of 1 at the origin. Notice the colors in the contour plot are reversed from those of Figure 1.

Finding Local Max's and Min's

Recall that the gradient vector ∇f points in the direction of greatest increase. If a function has a local maximum at the point (a, b), then there is *no* direction of greatest increase at (a, b) because the function is already at a maximum. Thus, we must have $\nabla f(a, b) = \mathbf{0}$. Similarly, if f has a local minimum at (a, b), then there is no direction that will decrease the function. Hence $-\nabla f(a, b)$, which points in the direction of greatest decrease, must vanish. This motivates the following definition.

Definition: A point (a, b) is called a **critical point** for $f(x, y)$ if $\nabla f(a, b) = \mathbf{0}$ or if $\nabla f(a, b)$ is undefined. A local max or min must always be a critical point.

Note that this generalizes the definition of a critical point for a function of one variable. To find a critical point of $f(x, y)$, we must solve the equations $f_x = 0$ and $f_y = 0$ *simultaneously.* For example, the function $f(x, y) = x^2 + y^2$ has $f_x = 2x$ and $f_y = 2y$. Solving $2x = 0$ and $2y = 0$ simultaneously gives $x = 0$ and $y = 0$. Thus $(x, y) = (0, 0)$ is the only critical point for f, as expected.

Exercise 1: Find the critical points of $f(x, y) = \sqrt{x^2 + y^2}$. Classify each one as a local or absolute min or max.

Example 3: $h(x, y) = x^2 - y^2$

The graph of h is a saddle and the only critical point is at the origin since $\nabla h = < 2x, -2y >= \mathbf{0}$ holds only if $x = y = 0$. However, the origin is neither a local min nor a local max because there are points arbitrarily close to the origin with function values greater than $h(0, 0) = 0$ and other points with function values less than 0 (notice the different colors in the contour plot in Figure 3 near the origin). A critical point with this property is called a **saddle point.**

Definition: The point (a, b) is called a **saddle point** if it is a critical point and if there are points (x_1, y_1) and (x_2, y_2) arbitrarily close to (a, b) such that $f(x_1, y_1) < f(a, b) < f(x_2, y_2)$.

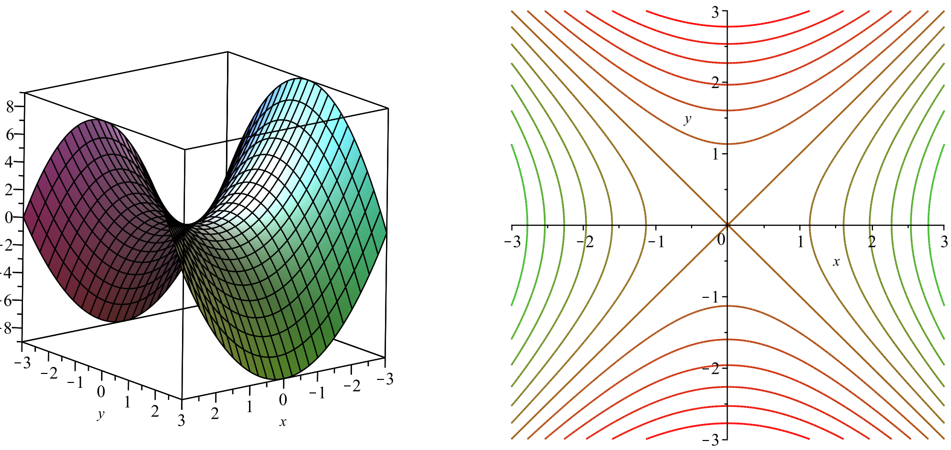

Figure 3: The function $h(x, y) = x^2 - y^2$ has a saddle point at the origin because $(0, 0)$ is a critical point and any neighborhood of the origin contains points with function values above and below $h(0, 0) = 0$.

Important Fact: Notice the difference between the contour plot about a local max or min (elliptical contours), and the contour plot near a saddle point (hyperbolic contours with an X through the saddle point). These contour plots are typical for these types of critical points.

The Second Derivative Test

We now learn the analog of the second derivative test used to determine the type of critical point.

Second Derivative Test: Let (a, b) be a critical point of $f(x, y)$. The matrix of second partials $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ evaluated at (a, b) is called the **Hessian matrix**. Let $D = f_{xx}f_{yy} - f_{xy}^2$ be the determinant of this matrix. Then we have

(i) if $D > 0$ and $f_{xx} > 0$, then (a, b) is a local minimum;

(ii) if $D > 0$ and $f_{xx} < 0$, then (a, b) is a local maximum;

(iii) if $D < 0$, then (a, b) is a saddle point;

(iv) if $D = 0$, the test is inconclusive.

Exercise 2: Using the second derivative test, check that the origin is a local minimum for $f(x, y) = x^2 + y^2$, a local maximum for $g(x, y) = 1 - x^2 - y^2$, and a saddle point for $h(x, y) = x^2 - y^2$.

Note that if $f_{xx} > 0$ and $f_{yy} > 0$ (so f is concave up in the x- and y-directions), it does not necessarily follow that the critical point is a local minimum. This is demonstrated by the next exercise.

Exercise 3: Check that the origin is a saddle point for $f(x, y) = x^2 - 3xy + y^2$, even though f_{xx} and f_{yy} are both positive.

Exercise 4: Consider the function $g(x, y) = y^3 - 12y + 2x^2 + 4x + 4$ from computer project #1. Find the critical points of g and classify them as local min's, max's or saddle points.

Exercise 5: Find the shortest distance from the point $P(2, -1, 0)$ to the plane $2x + y + z = 9$.

Hint: Let (x, y, z) be a point Q on the plane and let d be the distance between Q and P. Write $d = d(x, y)$ as a function of two variables and find the minimum of d^2.

Absolute Max's and Min's

One important fact about continuous functions is that they attain an absolute max and an absolute min on a closed and bounded region of $\mathbb{R}^n$. A set is **closed** if it contains all of its limit points (e.g., the boundary is included in the set). A set is **bounded** if it is contained in a disk, that is, the set does not head off toward infinity in any direction. To find the absolute max or min of a function on a closed and bounded set S, first locate all the critical points in S. Then search for extrema on the boundary of S by substituting the boundary expression(s) into the function. Compare the function values of all the points located to find the absolute max and min.

Exercise 6: Find the absolute maximum and absolute minimum for $f(x, y) = 4 - x^2 - y^4 + \frac{1}{2}y^2$ over the solid unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$.

Hint: First look for critical points in the interior of D. Then solve $x^2 + y^2 = 1$ for one of the variables and substitute into f to obtain a function of one variable. Find the extrema of this new, reduced function and compare their function values with those of the critical points in the interior.