MATH 241-02, Multivariable Calculus, Spring 2019

Section 11.1: Functions of Several Variables: Graphs and Contour Plots

In Chapter 11 we apply ideas from calculus to functions of several variables. We will learn about partial derivatives, tangent planes, the directional derivative, critical points, and Lagrange multipliers. We begin by visualizing functions of more than one variable using graphs and contour plots. Further exploration of these ideas will occur with computer project #1.

Functions of Two Variables

Recall that $z = f(x, y)$ is a function with two input variables (x, y) and one output variable z. The **graph** of a function of two variables lives in three dimensions (xyz-space). If $(0, -3, 4)$ is a point on the graph of f, then we have $f(0, -3) = 4$. The simplest way to think of a graph is regarding the xy-plane as where the domain of the function resides while the height of the graph represents the function value. Below is the graph of the function $f(x, y) = 7xy/e^{x^2+y^2}$. Notice that there seem to be two local maxima and two local minima.

Another important way to visualize a function is by drawing a **contour plot**. This is obtained by taking different traces $z = k$ of the graph and projecting them into the xy-plane. While the graph of $f(x, y)$ is three-dimensional, the contour plot lives in the **plane**. The contour plot for $f(x, y) = 7xy/e^{x^2+y^2}$ is shown to the right in Figure 1.

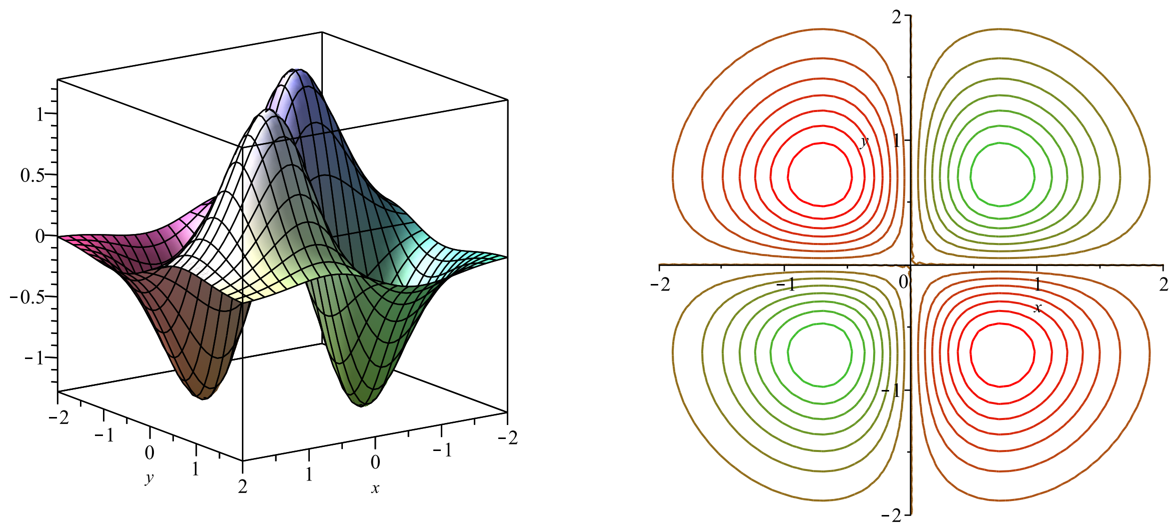

Figure 1: The graph of $f(x, y) = 7xy/e^{x^2+y^2}$ (left) and a contour plot of the same function (right).

The curves in the xy-plane of a contour plot are called **level curves** because they represent different "levels" of the function. All the points on one curve in a contour plot have the same function value (the same height on the graph). Thus a contour plot is a projection of different levels of a graph onto the xy-plane. In Figure 1, the red curves correspond to lower function values than the green curves. Can you see why this is the case based on the function? How does the contour plot help us locate the maxima and minima of the function?

Exercise 1: For the function $f(x, y) = 7xy/e^{x^2+y^2}$, what are the level curves for $z = 0$? In other words, what curve(s) in the xy-plane satisfy the equation $f(x, y) = 0$?

Important Convention: When making a contour plot, always use **equally spaced** values of z. For instance, we might choose $z = -2, -1.5, -1, -0.5, 0, 0.5, 1.0, 1.5, 2.0$ to obtain 9 equally spaced function values. However, the level curves themselves are **not** usually equally spaced!

Figure 2 shows the graph and contour plot of the function $g(x, y) = x^2 + y^2$. This function has an absolute minimum at $(0, 0)$ (the bottom of the bowl). The level curves $g(x, y) = k$ are concentric circles of radius $\sqrt{k}$ all centered at the origin. Notice that they are not equally spaced; as k increases, the circles get closer and closer together. Why?

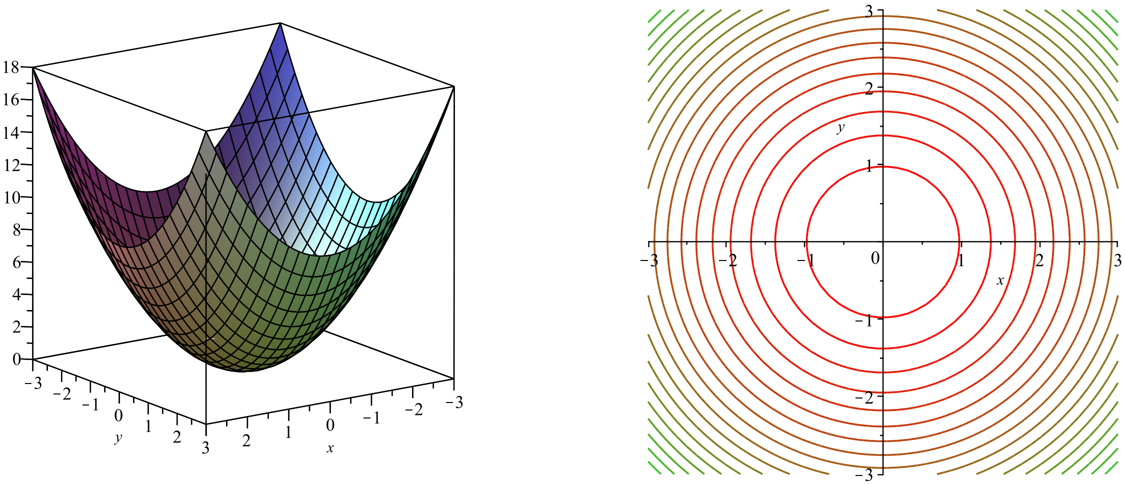

Figure 2: The graph of $g(x, y) = x^2 + y^2$ (left) is an elliptic paraboloid (bowl) and the contour plot (right) consists of circles getting closer together as the contour value increases.

Exercise 2: Sketch the contour plot for $f(x, y) = \sqrt{x^2 + y^2}$. What is the difference between this contour plot and the one for $g(x, y) = x^2 + y^2$? What is the graph of f?

Exercise 3: Sketch the contour plot for $f(x, y) = 2x + y - 3$. Are the contours equally spaced? What is the graph of this function?

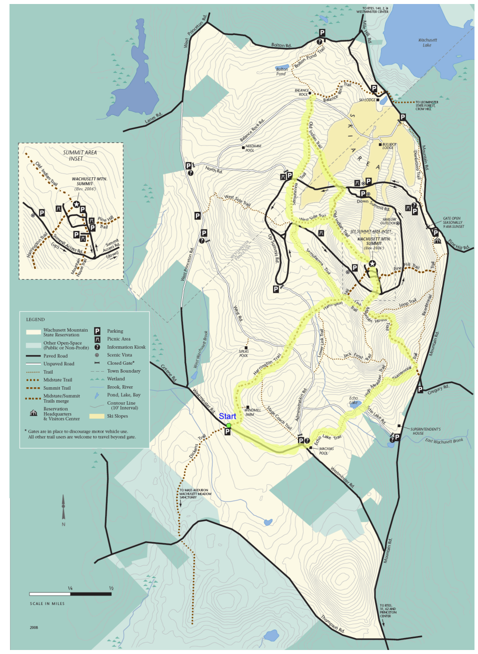

Figure 3: Contour plots: (left) rock formation on Jalama Beach, Santa Barbara, CA (©Bill Pollard, 2013); (right) topographic map of Mt. Wachusett.

Contour plots are useful visual tools in several applications. For example, a weather map showing different temperatures often includes level curves called **isothermals**. All of the locations on an isothermal have the **same** temperature. Two other examples are shown in Figure 3. For the rock formations on the left, larger rock sediments (glaciers?) formed these level curves over long periods of time. On the right is a topographic map for understanding the layout of Mt. Wachusett. Here the function captured is the height of the mountain, so locations on the same level curve have the same height. As contours get closer together, the mountain becomes more steep. The contour map provides essential information if you are establishing hiking trails.

Exercise 4: Sketch the contour plot for $f(x, y) = x^2 - y^2$ using the contour values $k = -2, -1, 0, 1, 2$. What is the graph of this function?

Exercise 5: Sketch and compare the contour plots for $f(x, y) = y - x$ and $g(x, y) = (y - x)^2$.

Key Points about Contour Plots

1. Every point on a particular level curve has the **same** function value.
2. Although the values chosen for a contour plot are equally spaced, this does **not** imply that the contours themselves are equally spaced.
3. Closely spaced contours indicate regions where the function is increasing (or decreasing) rapidly, while widely spaced contours correspond to locations where the function is increasing (or decreasing) slowly.
4. The set of allowable contour values is equivalent to the range of the function.

Level Surfaces

How do we visualize a function of three variables $w = f(x, y, z)$? Since the domain is three-dimensional, we would need **four** dimensions to draw the graph of the function ($xyzw$-space)! Instead, we focus on the contours, which are now **level surfaces** as opposed to level curves.

Example 1: Describe the level surfaces of $w = f(x, y, z) = x^2 + y^2 + z^2$.

Answer: Setting $w = k$, we have $x^2+y^2+z^2 = k$ as a level surface in xyz-space. These are concentric spheres centered at the origin of radius $\sqrt{k}$ (see Figure 4). If $k = 0$, then the sphere collapses to a point at the origin. The range of the function is $[0, \infty)$. As k increases, the spheres get closer together. Each point on a given sphere has the same function value.

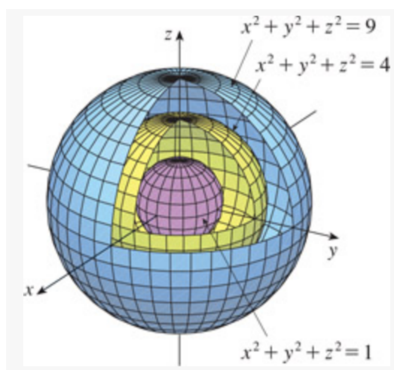

Figure 4: Level surfaces for the function $w = f(x, y, z) = x^2 + y^2 + z^2$ are concentric spheres.

Exercise 6: Describe the level surfaces of $w = g(x, y, z) = x^2 + y^2 - z^2$ for different values of $w = k$.