MATH 241-02, Multivariable Calculus, Spring 2019

Section 11.4: Tangent Planes and Linear Approximations

The goal of this section is to generalize the idea of the tangent line for functions of one variable to the tangent *plane* for functions of two variables. The tangent plane is the best linear approximation to a function.

Tangent Planes

Recall from Calc 1 that the equation of the tangent line to the function $y = f(x)$ at the point (x_0, y_0) is given by

$$y - y_0 \;=\; f'(x_0)(x - x_0)\,.$$

This is the point-slope form of a line with slope $m = f'(x_0)$, passing through the point (x_0, y_0). If f is differentiable at x_0, then as we zoom in on the graph of f near (x_0, y_0), the graph looks more and more like the tangent line.

We now generalize this same idea to a function of two variables, $z = f(x, y)$. Suppose that (x_0, y_0, z_0) is a point on the graph of f, that is, $z_0 = f(x_0, y_0)$, and suppose that both first partial derivatives f_x and f_y exist and are continuous at (x_0, y_0). Then the tangent plane to the graph of f at (x_0, y_0, z_0) is given by

$$\boxed{z - z_0 \;=\; f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)\,.} \tag{1}$$

The idea here is to capture the change in the function f in both the x-direction, through the term $f_x(x_0, y_0)(x - x_0)$, and the y-direction, via the term $f_y(x_0, y_0)(y - y_0)$. Notice that equation (1) is of the form $ax + by + cz = d$, so that it represents the equation of a plane, and that (x_0, y_0, z_0) satisfies the equation, so that it is a point on the plane.

Exercise 1: Use equation (1) to find the equation of the tangent planes to $f(x, y) = 1 - x^2 - y^2$ at the points **(a)** $(0, 0, 1)$ and **(b)** $(-2, 1, -4)$. Give a graphical explanation for your answer to **(a)**.

An Alternative Formula for the Tangent Plane:

Equation (1) can be rewritten as

$$\boxed{f_x(x_0, y_0)\, x + f_y(x_0, y_0)\, y - z \;=\; d\,,} \tag{2}$$

where d is a constant chosen so that (x_0, y_0, z_0) satisfies the equation. In other words, the tangent plane is the plane with normal vector $\mathbf{n} =< f_x, f_y, -1 >$ (evaluated at (x_0, y_0)) passing through the point (x_0, y_0, z_0). Formula (2) is a little easier to remember than equation (1). We will learn why the vector $\mathbf{n}$ is truly perpendicular to the graph of the function in Section 11.6.

Linear Approximation to $f(x, y)$ at (x_0, y_0)

One of the key ideas in Calc 1 is that the tangent line is the *best* linear approximation to a function. The same result holds for functions of two or more variables: **the tangent plane is the best linear approximation to a function.** The linearization is obtained by solving equation (1) for z and recalling that $z_0 = f(x_0, y_0)$. This gives

$$\boxed{L(x, y) \ = \ f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)\,.} \tag{3}$$

$L(x, y)$ is called the **linearization** or **linear approximation** of f at (x_0, y_0). Note that it is a linear function in the variables x and y, that is, L is of the form $L(x, y) = ax + by + c$ for some constants a, b, and c.

Exercise 2: Find the linearization for $f(x, y) = \sin(xy^2) + \sqrt{4x + y}$ about the point $(0, 1)$. Use it to estimate $f(-0.1, 1.05)$. Compare your estimate with the actual function value.

Differentiability

Recall from Calc 1 that a function $f(x)$ is differentiable at x_0 if $f'(x_0)$ exists. The definition of differentiability is more complicated for functions of two or more variables, but intuitively, we say that $z = f(x, y)$ is **differentiable** at (x_0, y_0) if the linear approximation is a good approximation for points near (x_0, y_0). In other words, differentiable functions are ones where the tangent plane approximates the function very well.

Example 1: Consider the functions $f(x, y) = x^2 + y^2$ and $g(x, y) = \sqrt{x^2 + y^2}$ near the origin $(0, 0)$. Both functions have global minima at $(0, 0, 0)$ (see Figure 1). The tangent plane for f at $(0, 0)$ is

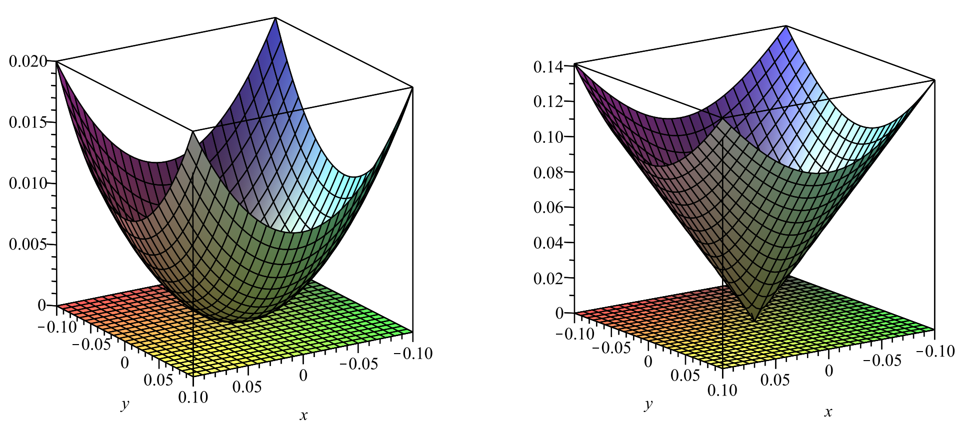

Figure 1: The graph of $f(x, y) = x^2 + y^2$ and $g(x, y) = \sqrt{x^2 + y^2}$ along with the plane $z = 0$.

simply $z = 0$ because $f_x(0,0) = 0$ and $f_y(0,0) = 0$. As we zoom in on the graph of f near the origin, it becomes flatter and flatter, and is well-approximated by its tangent plane. Thus f is differentiable at the origin.

On the other hand, the first partial derivatives of g do **not** exist at the origin. If we set $y = 0$, we have $g(x, 0) = \sqrt{x^2} = |x|$. Since $|x|$ is not differentiable at $x = 0$ (corner), $g_x(0,0)$ does not exist. A similar argument applies to $g_y(0,0)$. These facts are apparent in the graph of g near the vertex of the cone. No matter how much we zoom into the graph of g, there will always be a cone point; consequently, g is not well-approximated by the tangent plane $z = 0$, and is thus **not** differentiable at the origin.

The following fact is useful for determining whether a function is differentiable or not at a given point:

Useful Fact: If f_x and f_y exist and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Exercise 3: Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Use the limit definition of the partial derivative to show that $f_x(0,0) = f_y(0,0) = 0$. Conclude that the tangent plane for f at $(0,0)$ is $z = 0$. Is the function differentiable at $(0,0)$? Is it continuous at $(0,0)$?